**The Fundamental Theorem of Calculus**
**through the Eyes of Granville, Smith, and Longley**

I ‘d like to thank the usual suspects for allowing me to talk. I’d particularly like to thank David Bressoud, whose much more complete paper on the Fundamental Theorem will, I hope, soon be appearing in a journal near you. His comments on Granville, Smith, and Longley prompted my interest is this topic

What is the Fundamental Theorem of Calculus? There are two current contenders for this title, which are conceptually quite different:

Fundamental Theorem of Calculus 1. Suppose *f* is continuous on $[a,b]$. Then for any $x,\ a \le x \le b,\ \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

That is, a certain strangely defined function is differentiable and its derivative is something specified. Or, every continuous function has an antiderivative, namely, this strange function.

Fundamental Theorem of Calculus 2. Suppose *f* is continuous on $[a,b]$ and *F* is any antiderivative for *f* on that interval. Then $\int_a^b f(x)dx = F(b) - F(a)$.

That is, to calculate the integral of certain nice functions, all you need to do is find an antiderivative for the function.

I used the 1999 edition of Finney, Demana, Wiats, and Kennedy for these renditions of the Theorem(s), but I’m sure they don’t surprise you. If you’ve taught calculus any time in the last 50 years, this is at least in the text. Sometimes forms 1 and 2 are reversed; sometimes one is the theorem and the other is a corollary.

Generally, they are both there and at least one of them is called “The Fundamental Theorem of Calculus,” or perhaps ”The Fundamental Theorem of Integral Calculus.”

Now on the surface, these look different. One is about integrals and the other about derivatives. But Theorem 1 quite easily implies Theorem 2

Proof: 1 implies 2. Suppose *F* is any antiderivative of *f*. Then $\int_a^x f(t)dt = F(x) + C$ where *C* is some constant. (*Antiderivatives of the same function differ by a constant.*) But $0 = \int_a^a f(x)dx = F(a) + C$, or $C = -F(a)$. Substituting, we get 2.

So, let me ask you a question. What is fundamental about these theorems?
    “It connects the major operation of calculus.”

Indeed. What if I wrote "$\frac{d}{dx}\int f(x)dx = f(x)$"? Is that the same? It if hardly a surprising result: The derivative of any antiderivative is the original function.

"It allows one to compute integrals."

Of course, but life was not always sophisticated. That's why I want to go back to the beginning of the 20th century and look at a very popular calculus text.

Before I turn to my main subject, let me remind you of a bit of better-known history. One reason to think that version 2 is a spectacular theorem is that the definite integral is defined as a special kind of limit of sums. If you are like me, you have a difficult time keeping that idea in front of your students once they see the Fundamental Theorem. That integral can be defined this way originated with Cauchy in 1823 and was generalized by Riemann. In 1854.

Cauchy also proved version 1 in 1823.

We're about to see that none of this made any difference to Granville.

The text I'm going to following is that of Granville, Smith and Longley. It was originally published in 1905 and continued its run into the 1950s. Granville alone, then an instructor at Yale, wrote the first edition.

On page 287 of his 445-page book, Granville introduces his problem:

"The problems of the Integral Calculus depend on the *inverse operation*, namely:

*To find a function f(x) whose derivative*

(A) $f'(x) = \phi(x)$

*is given*.

…

The function f(x) thus found is called an integral of the given differential expression, the process of finding it is called integration, and the operation is indicated by writing the integral sign ∫ in front of the given differential expression. Thus,

(C) $\int f'(x)dx = f(x)$" pp. 287-288

In other words, for Granville, integration is anti-differentiation. In fact, he says in footnotes that this called an anti-differential by some authors and is sometimes denoted $D_x^{-1} f(x)$

About a half a page later, he emphasizes that integration and differentiation are inverse operation by writing "$d\int f'(x)dx = f'(x)$." Looks suspiciously like version 1.

Granville now introduces the constant of integration and proceeds to spend over 60 pages on techniques of integration.

On page 355, we get to a Chapter called "The Definite Integral." Granville begins by showing – not very well by modern standards – that if u gives the area under a curve $y = \phi(x)$, then $du = \phi(x)dx$ or, integrating, $u = \int \phi(x)dx$. Now, if $\int \phi(x)dx = f(x) + C$, then $u = f(x) + C$. Assuming the region in which interested goes from x =a to x = b, and following some calculations you've all seen before, We arrive at $area = f(b) - f(a)$. Following a brief recapitulation, Granville says, "We may accordingly define the symbol $\int_a^b \phi(x)dx$ as the numerical measure of the area bounded by curve $y = \phi(x)$, the axis of *X*, and the ordinates at *x* = *a*, *x* = *b*. … We have already shown that the numerical value of the definite integral is always $f(b) - f(a)$ [where f is an antiderivative of $\phi$]… .

So there you have it. NO Fundamental Theorem of Calculus. The following chapter is called "Integration a Process of Summation,' and in it, Granville shows that the definite integral is a limit of sums of products – which we would call "Riemann sums," and that this is a useful way of looking at it.

The next edition appeared in 1911, by which time Granville was the president of Gettysburg College and Percey Smith, who probably encouraged Granville to write the book in the first place, is listed "with the editorial cooperation of." The book is not very different from the 1905, but for purposes of this talk, there is a very big difference: A section called "The fundamental theorem of Integral Calculus." A section called "Analytical proof of the Fundamental Theorem" follows this. What do we have here?

"FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS. *Let ø(x) be continuous for the interval x = a to x = b. Let this interval be divided into n subintervals whose lengths are* $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$, *and points be chosen, one in each subinterval their abscissas being* $x_1, x_2, \cdots, x_n$, *respectively. Consider the sum*

*(C)* $$\phi(x_1)\Delta x_1 + \phi(x_2)\Delta x_2 + \cdots + \phi(x_n)\Delta x_n = \sum_{i=1}^{n} \phi(x_i)\Delta x_i.$$

*Then the limiting value of this sum when n increases without limit equals the value of the definite integral* $\int_a^b \phi(x)dx$. [p.363]

No, No, No! That's not the Fundamental Theorem; that's the *definition* of the definite integral! Not for Granville and Smith it isn't. The proof, by the way, is a very nice one depending on the Mean Value Theorem for derivatives, with only one fudge, which the authors acknowledge.

The 1929 version of the book was a substantial revision. By this time Granville had moved into the insurance industry in Chicago, and Smith needed a new junior colleague to revise the book. His choice was William R. Longley, a Chicago Ph.D. in astronomy and a far better mathematician than Granville. The language of the book was "modernized" and topics were added and subtracted. What hasn't changed is the Fundamental Theorem of Integral Calculus.

The book was revised again in 1933, 1941, and 1946. The last, by Longley alone. The market was still there; indeed, the book was used well into the 50s. But they were merely reprintings of the earlier editions.

The recognition that what I had assumed was the fundamental theorem was not at all in a major text of the first half of the 20th century sent me scrambling to my own education. I learned (but just barely) calculus at Oberlin College in 1955—56. The text I used was by Ed Begle. It was very rigorous and the theorems we would expect were there. The second form of the Fundamental Theorem was called that. My girl friend, now my wife, who learned calculus at the same time in the same place, used a text by Lloyd Smail, copyrighted 1953. He, too, defined the integral as he limit of a sum and proved the second form of the Fundamental Theorem. The next year, I used a text by Sherwood and Taylor. They proved both forms of the Fundamental Theorem, but labeled neither.

What should we learn from this?

1. The teaching of calculus is fashionable. By that, I mean, that the way we do it now is not the only way, but may not have been the way it was done quite recently.

2. We need to be cautious when we refer to a theorem as "The Fundamental Theorem of this or that." Its central role may not be apparent to our students and may not have been apparent to our predecessors.

3. Even the theorem to which the name "Fundamental" is attached may change