History in a Math Course for Teachers

by

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Introduction:

When I teach math courses for teachers, I try to put history into the course for the usual reasons:

- To make mathematics more “human”
- To make mathematics more understandable
- To give teachers ideas and materials that can help them teach math more effectively.

However, achieving these goals within the context of a math course for teachers turns out to be an interesting challenge. This paper describes some of my efforts to meet that challenge.

The context

It is scandalous, on the face of it, that college credit is awarded for a mathematics course whose mathematical content is at a pre-college level. The college math for teachers course is a last chance to help prospective teachers overcome deficiencies in their mathematical training so serious as to severely threaten their ability to teach K-8 mathematics effectively. These deficiencies are all the more shocking in view of the many years of mathematics courses that these prospective teachers have taken (or had administered to them) in their K-12 and college careers.

K-8 mathematics now includes, in addition to arithmetic, the algebra equivalent to algebra 1 (up to quadratic equations) as well as a little probability and statistics. Those topics determine the course content of math courses for teachers. Many of these students are math avoiders, who take math courses only if required to do so, and it has long been observed that teachers who dislike mathematics and are not very good at it are likely to produce students of their own who dislike mathematics and are not very good at it. Thus, the math course for teachers is an opportunity to break a very unfortunate cycle.

In designing the course, it is important to:

Meet students where they are, and build on their strengths. (Even the weakest have learned something in all those years.)

Insist that they master the material in depth, while minimizing the pressure of tests and quizzes.
Present the material in ways that can be adapted to K-8 classrooms.

A historical or a "genetic" approach in the style of Toeplitz[1] or Bressoud[2] does not fit this course, because the order in which topics must be presented differs radically from the historical order of their development. Nevertheless, there are many ways to use history profitably in this course, as is shown below.

Small (but useful) uses of history

A first step is to include historical information, often in the form of footnotes, explaining the origin of symbols (e.g. the plus sign's origin as a hasty "et") and telling something of the people and contexts that gave rise to the ideas being presented. (Examples are given below. This entire article, including the examples, is based on the book noted at the end of this article). A similar step is to use problems and examples of historical interest. Here are a couple of them.

A certain man drinks a barrel of beer in 20 days. If his wife drinks with him, they will empty the barrel in 14 days. How long would the woman take to drink the barrel alone? (Adapted from Peter van Halle's arithmetic text of 1568[3])

Some thieves, who had stolen silk cloth, gathered together under a bridge to decide how to distribute their ill-gotten gains among them. Some pedestrians who happened to pass the bridge overheard their loud arguments. They were saying that if each one took 12 *tan*, then 12 *tan* would be left, and if each

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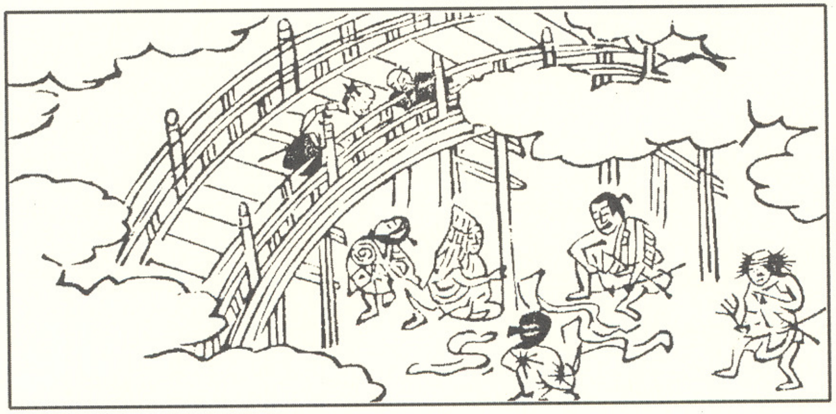
[1] Otto Toeplit. *Calculus, a Genetic Aprroach.* Translated from the German byLouise Lange. University of Chicago Press, 1963.

[2] David Bressoud, *A Radical Approach to Real Analyis*, Mathematical Association of America, Washington, DC, 1994.

[3] Problem 5 is from Marjolein Kool, *Two Pots with One Lid: The first arithmetic textbooks in the Netherlands before 1600*, *Proceedings of the HPM Conference on History in Mathematics Education*, Department of Mathematics, National Taiwan Normal University, 2000.



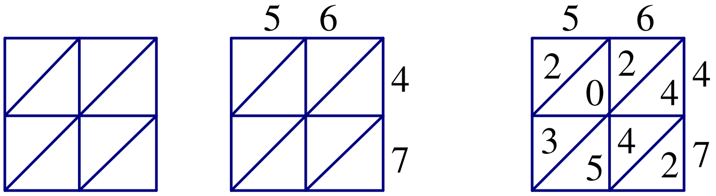
one took 14 *tan*, then 6 *tan* would be short. How many thieves were there and how much cloth was there?[4]

Individually, these footnotes and examples have little importance, but a lot of them, collectively, can significantly influence the tone of the course.

A step up in importance are exercises with a historical focus that directly engage students in new ways to think about the mathematics they are to learn and teach. Here are some examples.

Lattice (or Gelosia) Multiplication

This method was widely used in Europe during the renaissance. To find 56x47, make a 2x2 array (since each factor has 2 digits), with each box cut diagonally as in figure 1.4.9a, then write the factors at the top and right as in figure 1.4.9b.



1.4.9a□□□□□□ □□□□1.4.9b□□□□□□□□□

Figures 1.4.9a,b, c

Next, multiply as you would in a box puzzle, but write units digits below the diagonals and the tens digits above them, as in figure 1.4.9c. Finally, add along the diagonals, starting with the 2 in the lower right-hand corner. The next diagonal has 4+4+5, or 13, from which the 3 is written, and the 1 is carried into the next diagonal addition 2+0+3 to get 6 in all. Finally, the 2 at the upper left has nothing to add to it. The product 56x47=2,632 is blue in figure 1.4.9d.

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[4] Adapted, with permission,from *Jinkoki*, November, 1641 edition, translated into English by Osamu Takenouchi and published by Wasan Institute, 2000. *Jinkoki* is representative of the "mathematics for ordinary people" developed in Japan in the period 1603-1867.

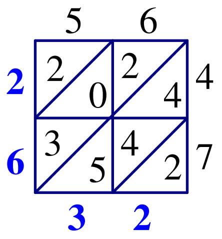

Figure 1.4.9d

Successive Doubling This method is described in an Egyptian papyrus from around 1650 B.C.E. Again consider 56x47. Regard the problem as one of putting together a large number of 47's, and begin by doubling to find:

1x47=47
2x47=94
4x47=188
8x47=376
16x47=752
32x47=1,504

Stop here, since the next step would be 64x47, which is more 47s than wanted. Since 56=32+16+8, add the corresponding multiples: 56x47=1,504+752+376=2,632. This doubling method is slow, but it might help students who are having trouble learning basic multiplication facts.

An Exploration: the Sieve of Eratosthenes[5] The sieve of Eratosthenes is a way to find all the primes less than a given number. To illustrate, we use it to find all primes less than 100. Begin with the table used in exploring multiples and follow the steps (parts a-h).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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[5] Eratosthenes of Cyrene (about 274-194 B.C.E.) taught in Alexandria after about 240 B.C.E., where he was librarian. His main interests were poetry and geography, but he is also remembered in mathematics for his "sieve" for determining all the prime numbers up to any given size. His nickname was "Beta."

a. The smallest prime is 2. Since each higher multiple of 2 is composite (cannot possibly be a prime), cross them all out. The primes less than 100 are among the numbers **not** crossed out.
b. The next smallest prime is 3, since it is not a multiple of any smaller prime. Using a different color pencil, cross out every multiple of 3 except for 3 itself.
c. The next smallest prime is 5. Using a third color, cross out all higher multiples of 5 i.e., 10, 15, 20, ... (This problem continues. . .)

An exercise in reducing fractions

a. Rewrite this table with each fraction reduced to lowest terms.

Row number	**Column number**				
	0	**1**	**2**	**3**	**4**
0	1	$\frac{1}{1}$	$\frac{1\cdot 2}{1\cdot 2}$	$\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3}$	$\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4}$
1	1	$\frac{2}{1}$	$\frac{2\cdot 3}{1\cdot 2}$	$\frac{2\cdot 3\cdot 4}{1\cdot 2\cdot 3}$	$\frac{2\cdot 3\cdot 4\cdot 5}{1\cdot 2\cdot 3\cdot 4}$
2	1	$\frac{3}{1}$	$\frac{3\cdot 4}{1\cdot 2}$	$\frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}$	$\frac{3\cdot 4\cdot 5\cdot 6}{1\cdot 2\cdot 3\cdot 4}$
3	1	$\frac{4}{1}$	$\frac{4\cdot 5}{1\cdot 2}$	$\frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}$	$\frac{4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4}$
4	1	$\frac{5}{1}$	$\frac{5\cdot 6}{1\cdot 2}$	$\frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}$	$\frac{5\cdot 6\cdot 7\cdot 8}{1\cdot 2\cdot 3\cdot 4}$

b. Use the patterns to extend the table one more row down and one more column to the right. This pattern was discovered by Thomas Harriot.[6] Where have you seen it before?

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[6] Thomas Harriot (1560-1621) of Oxfordshire, graduated from Oxford in 1577 and then moved to London. Sir Walter Raleigh sent him to Virginia in 1585 as navigator. While in Virginia (today North Carolina) Harriot studied the local flora and fauna and invented a remarkable written alphabet for a local Native American language. On returning to England, he published his *True Report of the Newfound Land of Virginia* (1588) and continued to work on science and mathematics for his patrons, Raleigh and Henry Percy. Queen Elizabeth's successor James I imprisoned Raleigh in the Tower of London, and Harriot, too, was imprisoned there briefly in 1605. Harriot, who had learned to smoke tobacco in Virginia, died of cancer of the nose, having made extensive but largely unpublished contributions in mathematics and science. He introduced the convention that if two numbers are written next to each other and no other operation is indicate, they are to be multiplied. He also introduced the inequality signs < and >.

Sometimes it is useful to insert history into the exposition itself. Here are two examples.

The bias against negative numbers

Since people first began to count, the concept of number has been repeatedly extended and refined. One such extension is to numbers "below zero." Numbers below zero are known as **negative** numbers, a name with unfortunate connotations. (A negative attitude, for example, is generally considered bad.) The medieval church, which associated negative numbers with debt and the sin of usury, regarded those numbers as sinful. This effectively banned negative numbers until the Renaissance, and even then they were written in the devil's color, red, a tradition that endures in modern bookkeeping.[7]

Numeration systems and powers

Our modern decimal system of numeration is believed to have originated in India between 300 B.C.E. and 800 C.E. The details are not known today, but the *Aryabhatiya*, a 6th century book of verses summarizing mathematics of the time by Aryabhata, contains the maxim, "**From place to place each is ten times the preceding**," which certainly sounds as if he had the idea. Over a century later, a book by Brahmagupta specifically used zero and negative numbers. When the caliph al-Mamun (809-833) established a "House of Wisdom" in Baghdad, where translating books from other cultures into Arabic was a major activity, the mathematician and astronomer al-Khwarizmi wrote *Concerning the Hindu Art of Reckoning* to introduce Hindu numeration to the Arab world. Over 300 years later, Leonardo of Pisa[8] learned Hindu numeration from Arabs and wrote the *Liber Abaci* (*Book of Reckoning*) in1202 to explain that numeration to Europeans. At the time, Europe still used Roman numerals, which are so clumsy that in the middle ages the few people who could do arithmetic without an abacus or "counting board" were sometimes suspected of supernatural powers.

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[7] Negative numbers were used in China some 2,000 years ago, and they were also used in India for several centuries before they were accepted in Europe. The Alexandrian Greek mathematician Diophantus, thought to have lived around 250 C.E., used negative numbers quite extensively, but the level of his work with algebra and number theory, and the use of negative numbers, were not attained again in Europe for more than a millennium.

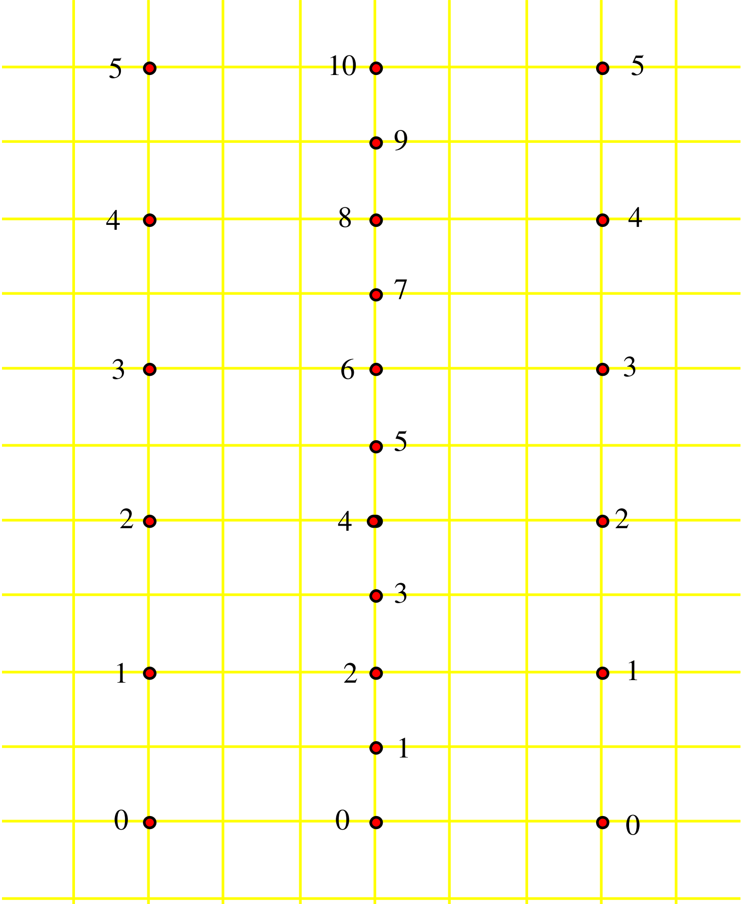
[8] Leonardo Pisano (ca. 1170-1250) was given the nickname, Fibonacci, which means "Son of Bonaccio" by a 19th century editor of his work. (Leonardo's father was named Guilielmi; Bonacci may have been another ancestor, such as a grandfather.) Born in Pisa, he learned Hindu notation in Bugia, in what is today Algeria, where his father was sent with Pisan traders. Later he traveled throughout the Eastern Mediterranean, studying Islamic mathematics. Other Europeans had used decimal numeration. (One, Gerbert D'Aurillac, who later became Pope, had done so over 200 years earlier, but he had only crude methods for calculating with the new numerals.) The *Liber Abaci* did much to popularize decimal numeration. Fibonacci is best known today for the so-called Fibonacci numbers (see Exercises 5.1).

Large Uses of History

In contrast to isolated examples, be they footnotes, exercises, or text, are uses of history that play a major role in the presentation of the mathematics itself. Here are two examples.

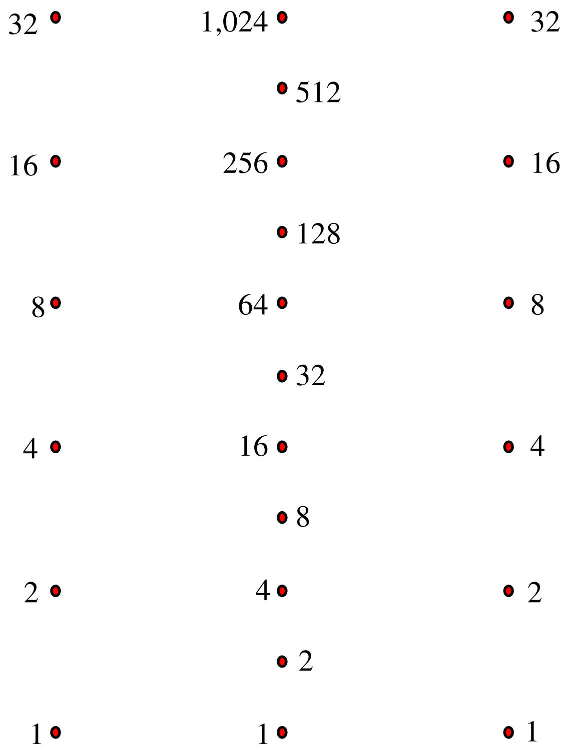
A Historically Informed Treatment of Rational Exponents

1. Make an adder (This is actually done earlier, near beginning of book).

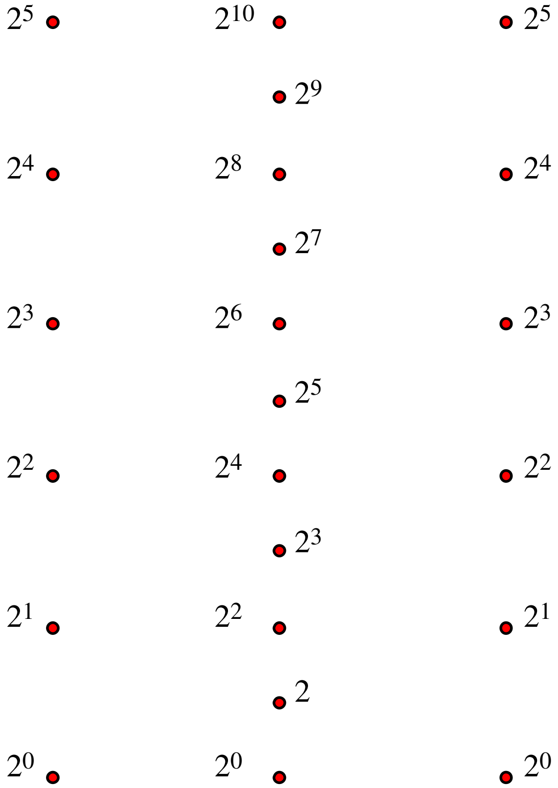


What numbers does it work for? Why does it work?

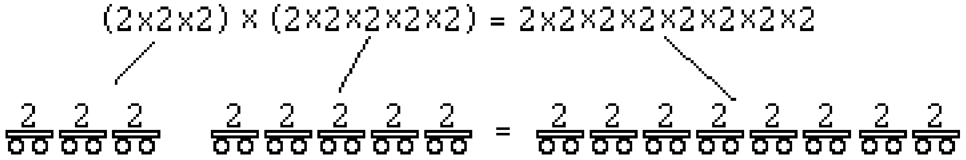
Now ask: Could you make a multiplier using the same dot pattern?
1 plays the role for multiplication that 0 plays for addition. (an identity), so start with a row of dots numbered 1. If you put a 2 on the next dot above, what happens?



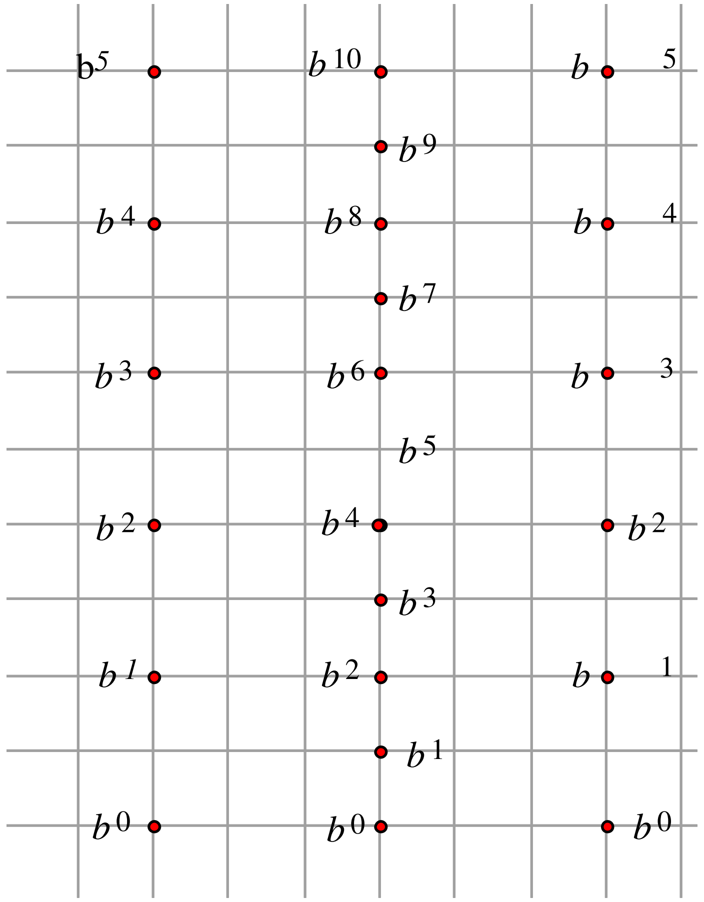
Critique the multiplier. What does it do? What does it not do? How does it work?



This is, in effect, an adder for exponents! This is easy to understand, if you think what the exponents mean. If you multiply 2^3 by 2^5, for example, think of 2^3 and 2^5 as trains of 2s, one with three cars, the other with five. Multiplying these numbers is like hooking the trains together to form a single train, which of course has 3+5=8 cars, as in figure 17.1.11.



Two two trains (make one big two train)
Figure 17.1.11



What if you use a value of *b* close to 1? Here is a table with *b*=1.1

10.83	117.39	10.83
	106.72	
9.85	97.02	9.85
	88.20	
8.95	80.18	8.95
	72.89	
8.14	66.26	8.14
	60.24	
7.40	54.76	7.40
	49.79	
6.73	45.26	6.73
	41.14	
6.12	37.40	6.12
	34.00	
5.56	30.91	5.56
	28.10	
5.05	25.55	5.05
	23.23	
4.59	21.11	4.59
	19.19	
4.18	17.45	4.18
	15.86	
3.80	14.42	3.80
	13.11	
3.45	11.92	3.45
	10.83	
3.14	9.85	3.14
	8.95	
2.85	8.14	2.85
	7.40	
2.59	6.73	2.59
	6.12	
2.36	5.56	2.36
	5.05	
2.14	4.59	2.14
	4.18	
1.95	3.80	1.95
	3.45	
1.77	3.14	1.77
	2.85	
1.61	2.59	1.61
	2.36	
1.46	2.14	1.46
	1.95	
1.33	1.77	1.33
	1.61	
1.21	1.46	1.21
	1.33	
1.10	1.21	1.10
	1.10	
1.00	1.00	1.00

Figure 17.2.4

This device is quite practical. Test I for accuracy. How would you improve it further?

n	**Approximate value of 1.1^n**	**n**	**Approximate value of 1.1^n**	**n**	**Approximate value of 1.1^n**
1	1.1	18	5.5599	35	28.1024
2	1.21	19	6.1159	36	30.9127
3	1.331	20	6.7275	37	34.0039
4	1.4641	21	7.4002	38	37.4043
5	1.6105	22	8.1403	39	41.1448
6	1.7716	23	8.9543	40	45.2593
7	1.9487	24	9.8497	41	49.7852
8	2.1436	25	10.8347	42	54.7637
9	2.3579	26	11.9182	43	60.2401
10	2.5937	27	13.1100	44	66.2641
11	2.8531	28	14.4210	45	72.8905
12	3.1384	29	15.8631	46	80.1795
13	3.4523	30	17.4494	47	88.1975
14	3.7975	31	19.1943	48	97.0172
15	4.1772	32	21.1138	49	106.7190
16	4.5950	33	23.2252	50	117.3909
17	5.0545	34	25.5477		

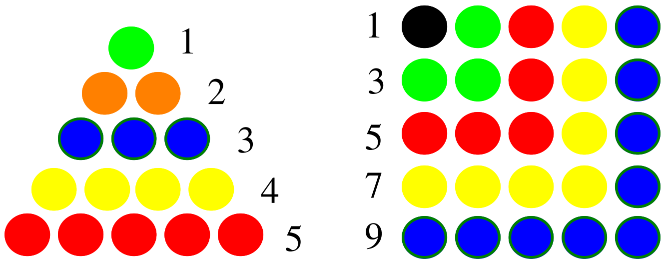
Note how handy this table is for powers and roots. It just begs for fraction exponents, to say nothing of scientific notation!

Though historically informed, this is much more sensible than the way the subject really developed. By learning history, we can avoid repeating it.

Figurate Numbers

This topic was of prime importance to the Pythagoreans and to many who followed them, but it has been almost crowed out of modern curricula. But figurate numbers are a good topic for a math course for teachers:

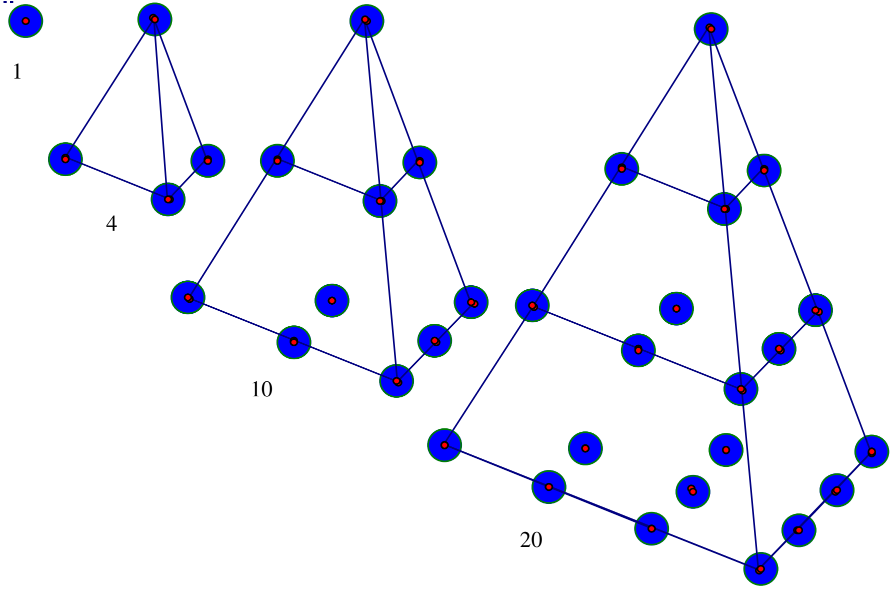
- They involve relatively simple whole numbers.
- They give many opportunities to observe in concrete situations and then move to an abstract level.



Other examples:

- The sum of consecutive triangular numbers as a square
- The odd squares as one more than 8 times a triangular number

Generalized to 3 dimensions, this yields cubes and tetrahedral numbers..



Early treatment of figurate numbers influences the rest of the course, making possible examples and exercises that would otherwise be skipped.

Applications of figurate numbers to the multiplication table.

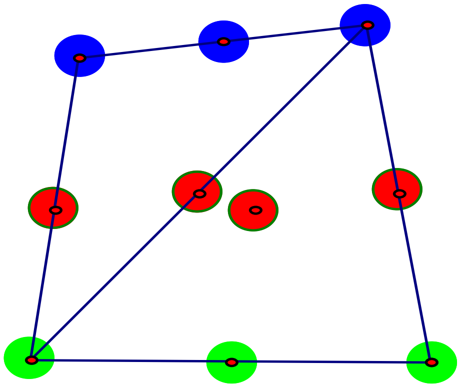
1. Differences of squares (Describe how this is used)

10	20	30	40	50	60	70	80	90	100
9	18	27	36	45	54	63	72	81	90
8	16	24	32	40	48	56	64	72	80
7	14	21	28	35	42	49	56	63	70
6	12	18	24	30	36	42	48	54	60
5	10	15	20	25	30	35	40	45	50
4	8	12	16	20	24	28	32	36	40
3	6	9	12	15	18	21	24	27	30
2	4	6	8	10	12	14	16	18	20
1	2	3	4	5	6	7	8	9	10

2. Tetrahedral numbers in the multiplication table.

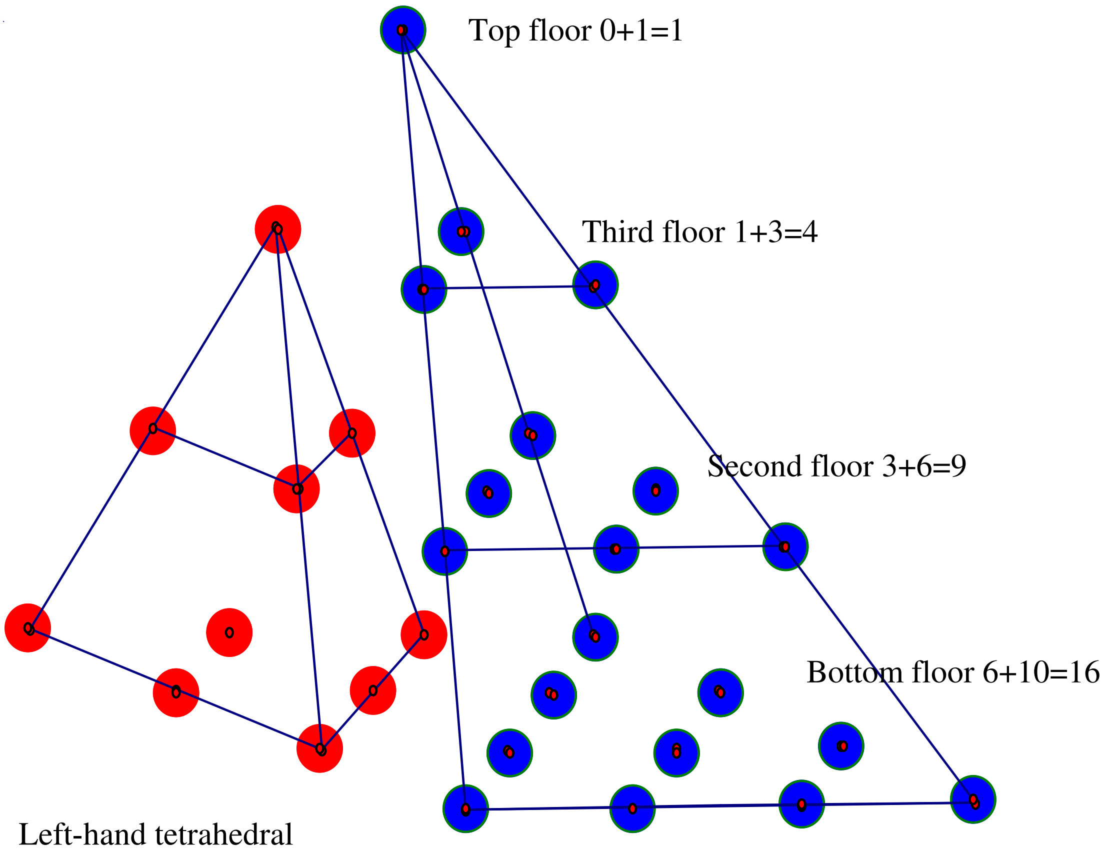
10	20	30	40	50	60	70	80	90	100
9	18	27	36	45	54	63	72	81	90
8	16	24	32	40	48	56	64	72	80
7	14	21	28	35	42	49	56	63	70
6	12	18	24	30	36	42	48	54	60
5	10	15	20	25	30	35	40	45	50
4	8	12	16	20	24	28	32	36	40
3	6	9	12	15	18	21	24	27	30
2	4	6	8	10	12	14	16	18	20
1	2	3	4	5	6	7	8	9	10

Why does this happen? Here's an "aha" proof for that



Treating figurate numbers early makes possible interesting exercises and insights that would otherwise be missed. Here are further examples.

We have seen the sum of the first *n* integers is a triangular number. The sum of the first *n* squares is usually postponed until algebra 2, when it is treated (boringly) by mathematical induction. Having studied figurate numbers, we can do better.



Left-hand tetrahedral number:
$10=1+(3+6)=1^2+3^2$

Right-hand tetrahedral number:
$20=(1+3)+(6+10)=2^2+4^2$

What about the sum of the first *n* cubes?

A series of problems on counting the number of rectangles in an $n \times n$ ruled square shows the students that this number can be expressed either as the sum of cubes of the numbers from 1 to *n* or as the sum of the numbers in a square corner of the multiplication table. That was shown to be the case by the Arab writer Al Karaji (died, 1019). Here is the idea.

a. An L-shaped part of the multiplication table is $\begin{matrix}2 & 4\\ & 2\end{matrix}$. These numbers are pictured in figure 4.1.3

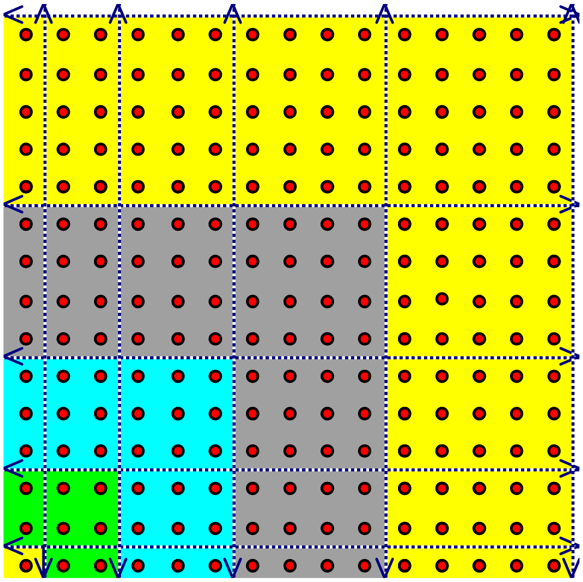
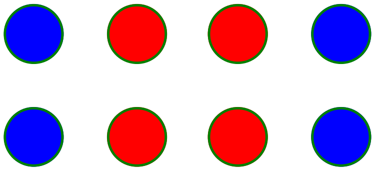
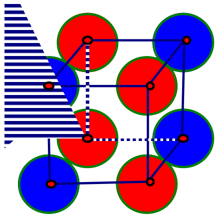


Figure 4.1.3

as well as below. To see these numbers add up to the cube 8, rearrange the arrays below as a cube:

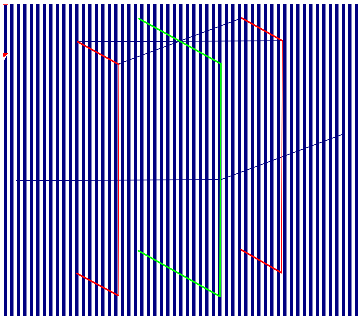




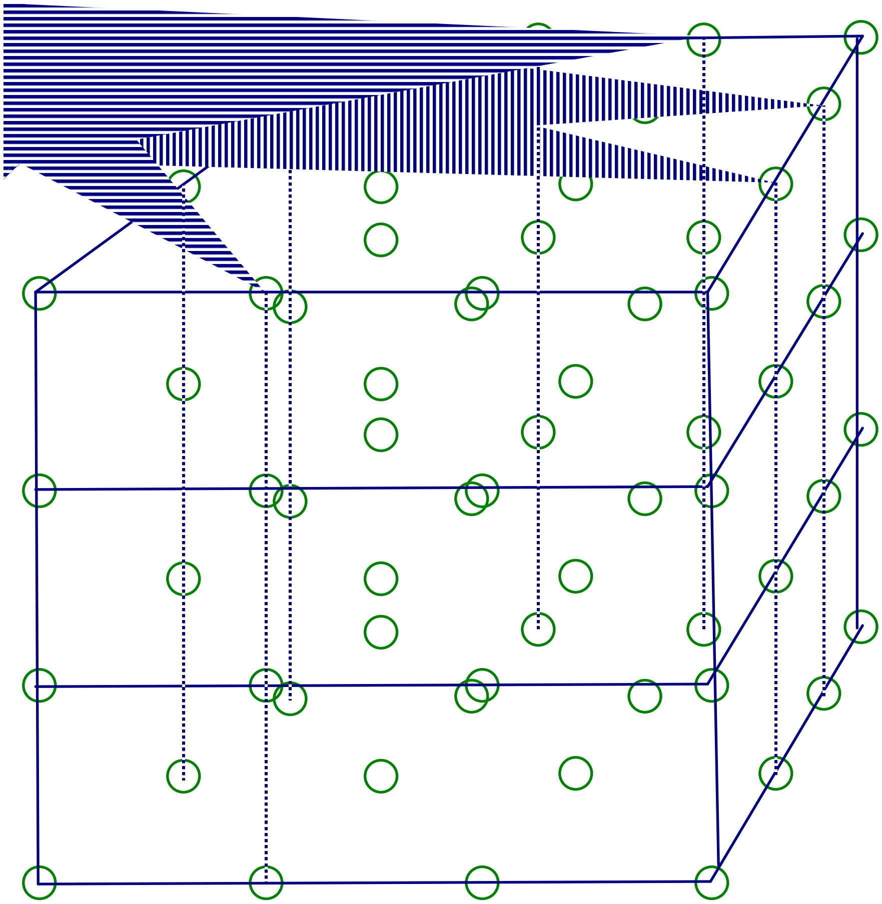
The L-shaped part of the multiplication table that borders $\begin{matrix}2 & 4\\ & 2\end{matrix}$ is

$$\begin{matrix}3 & 6 & 9\\ & & 6\\ & & 3\end{matrix}$$

. Explain why these numbers, 3, 6, 9, 6, 3 add up to the cube $3\cdot 3\cdot 3$. Use both figure 4.1.3 and the picture of the cube $3\cdot 3\cdot 3$ below in your explanation.



b. Color the $4\cdot 4\cdot 4$ cube shown below analogously to the cube in part a, to show that $1\cdot 4+2\cdot 4+3\cdot 4+4\cdot 4+3\cdot 4+2\cdot 4+1\cdot 4=4\cdot 4\cdot 4$.



Putting in the historical topic of figurate numbers enriches the rest of the text. Here is an exploration from the chapter on polynomials, that would not be possible had not figurate numbers been treated early on**.**

35. In section 3.1 it was seen that the sum of the first *n* whole numbers is the triangular number $\frac{n(n+1)}{2}$, which may also be written as a polynomial, $\frac{1}{2}n^2 + \frac{1}{2}n$. Verify this formula for the cases:

a. n=7 b. n=100

36. Problems 9 and 10 of Exercises 3.2 lead to a formula for the sum of the first *n* squares, which may be written as the polynomial $\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. Verify this formula for the cases:

a. n=3 b. n=5 c. n=6

37. Questions 15-21 of Exercises 5.1 involve the pattern $1^3+2^3+3^3+\cdots+n^3=(1+2+3+\cdots+n)^2$. Using the result of question 1 above, this leads to the formula $1^3+2^3+3^3+\cdots+n^3=\left[\frac{n(n+1)}{2}\right]^2$.

a. Show that the above expression may be written as the polynomial $\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$.

b. Add the first seven cubes and show that the sum is the same number as the value of the polynomial in part a when n=7.

38. The results of questions 35--38 were all known in ancient times. About a thousand years ago the Arab mathematician Alhazen found a formula for the sum of the first n fourth powers, which can be written as the polynomial $\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3+0n^2-\frac{1}{30}n$. Check this formula for the cases n=:

a. 1 b. 2 c. 3 d. 4

17th century mathematicians tried to find formulas like those of questions 35-38 for higher powers. However, finding such formulas was a lot of work, and eventually one mathematician, Jakob Bernoulli[9], partly in hopes of easing this work, compiled a table of the formulas for the first kth powers, for k=1, 2, 3, . . , 10, and looked for patterns in the table. His table is reproduced below. Here the symbol $S_6(n)$, for example, is the formula for $1^6+2^6+3^6+\cdots+n^6$. Explore this table on your own, looking for patterns. Then try the exercises below, which will guide you in further exploration.

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[9] Jacob Bernoulli (1654-1705), first of a family of outstanding mathematicians and scientists, wrote his *Ars Conjectandi (The Art of the Conjecture),* which included the work above on sums of powers, The *Ars Conjectandi* effectively established the probability as a branch of mathematics.

$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + 0n$$

$$S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 + 0n^2 - \frac{1}{30}n$$

$$S_5(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 + 0n^3 - \frac{1}{12}n^2 + 0n$$

$$S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 + 0n^4 - \frac{1}{6}n^3 + 0n^2 + \frac{1}{42}n$$

$$S_7(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 + 0n^5 - \frac{7}{24}n^4 + 0n^3 + \frac{1}{12}n^2 + 0n$$

$$S_8(n) = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 + 0n^6 - \frac{7}{15}n^5 + 0n^4 + \frac{2}{9}n^3 + 0n^2 - \frac{1}{30}n$$

$$S_9(n) = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 + 0n^7 - \frac{7}{10}n^6 + 0n^5 + \frac{1}{2}n^4 + 0n^3 - \frac{3}{20}n^2 + 0n$$

$$S_{10}(n) = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 + 0n^8 - 1n^7 + 0n^6 + 1n^5 + 0n^4 - \frac{1}{2}n^3 + 0n^2 + \frac{5}{66}n$$

39. Consider the numerical array of coefficients of the above polynomials:

$$\begin{array}{ccccccccccc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\
\frac{1}{5} & \frac{1}{2} & \frac{1}{3} & 0 & \frac{-1}{30} \\
\frac{1}{6} & \frac{1}{2} & \frac{5}{12} & 0 & \frac{-1}{12} & 0 \\
\frac{1}{7} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{6} & 0 & \frac{1}{42} \\
\frac{1}{8} & \frac{1}{2} & \frac{7}{12} & 0 & \frac{-7}{24} & 0 & \frac{1}{12} & 0 \\
\frac{1}{9} & \frac{1}{2} & \frac{2}{3} & 0 & \frac{-7}{15} & 0 & \frac{2}{9} & 0 & \frac{-1}{30} \\
\frac{1}{10} & \frac{1}{2} & \frac{3}{4} & 0 & \frac{-7}{10} & 0 & \frac{1}{2} & 0 & \frac{-3}{20} & 0 \\
\frac{1}{11} & \frac{1}{2} & \frac{5}{6} & 0 & -1 & 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{5}{66}
\end{array}$$

a. Extend the leftmost column down three more entries

b. Extend the second column (next to the left) down three more entries.

c. Which other columns can you extend?

40. a. Express each entry in the third column as a number of twelfths.
    b. Find a pattern in your answer to part a and use it to predict the next three entries of the third column.

41. a. Find the lowest common denominator of the fractions in the fifth column.
    b. Express each number in the fifth column as a fraction with the denominator found in part a.
    c. Find the pattern of the numerators of the fractions found in part b and use that pattern to predict the next three entries in the fifth column.

42. a. Add the fractions in each horizontal row. Show all the work.
    b. Based on your results from question 3, make a general statement about the sum of the entries in each row of this array.

Each of the formulas for the sum of first, second, and third powers is itself a significant generalization. The idea that those powers themselves can be generalized makes it possible to reach a level of abstraction not common in math courses for teachers.

Conclusion

I have presented just a few of the many ways in which history can be used in a math course for teachers. They and many more can be found in my book, Stein, R. G. with Wallace, L. ***Math for Teachers, An Exploratory Approach*** Kendall Hunt, Dubuque Iowa, 2009. Collectively, they use history to enliven, enrich, and deepen the course.