Does the Term "Dark Ages" Reflect Eurocentricism in Mathematics?

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Abstract

Although it might be true that there was an overall stagnation in the practice of mathematics and natural sciences in Europe between 600 and 1200, an examination of the works of some Chinese, Indian, and Middle Eastern scholars of the same period, such as Aryabhata I (475 - 550), Brahmagupta (598 - 670), Bhashkara I(600 – 680), Li Chunfeng (602 - 670), Al-Jawhari (800 – 860), Mahavira (800 - 870), Thabit (836 - 901), Abu Kamil (850 - 930), al-Battani (850 - 929), Sridhara (870 - 930), Aryabhata II (920 - 1000), Abu'l Wafa (940 - 998), al-Quhi (c. 940 - c. 1000), Al-Khujandi (c. 940 - c. 1000), Al-Haitam (965 - 1039), al-Biruni (973 - 1048), Avicenna (c. 980 - 1037), Al-Jayvani (989 - 1079), Omar Khayyam (1048 - 1122), Brahmadeva (1060-1130), Bhaskara II (1114 - 1185), Qin Jiushao (1202 – 1261), al-Maghribi (1220 - 1280), al-Samarqandi (1250 - 1310), al-Banna (1256 - 1321), and al-Farisi (1260 - 1320), shows that natural sciences, medicine, philosophy, astronomy, logic, biology, and mathematics were thriving in those regions. In particular, it was through the works of Mohammad Abu'l-Wafa Al-Buzjani, Abu Ja'far Mohammad ibn Musa Al-Khwarizmi, and Nasir al-Din al-Tusi, that several important concepts in modern plane trigonometry (such as constructing accurate tables of sine function and the use of the tangent function), spherical trigonometry (such as solving for parts of spherical triangles), the theory of equations (such as solutions of quadratic equations), and circular motion (cycloids), were either introduced or improved. It must be noted that these scholars, who were also prolific writers, made several vital contributions to the fields of astronomy, logic, physics, and biology as well.

In this paper I will give examples of the works of al-Buzjani, Al-Khawarizmi, and especially al-Tusi to show that in the seven centuries that span the period 800 - 1500 C.E., mathematical sciences were able to sail along in a relatively uninterrupted manner and even manage to flourish through the innovative endeavors of these and some other non-European scholars. Indeed by early twentieth century, many historians of science such as Bigourdan (1911), Cantor (1922), Doublet (1922), Boquet (1925), and Dilgan (1958) had begun to acknowledge this fact.

Keywords: Plane trigonometry, the sine function, the tangent function, spherical trigonometry, the theory of quadratic equations, cycloid, astronomy

1. Introduction

The early Middle Ages or Dark Ages that spanned nearly a millennia, from 300AD to 1100AD, witnessed less than modest progress in the field of mathematics in Europe. One possible reason for this stagnation was the centuries long domination of the European world by the Roman culture whose main concern, unlike its more philosophical minded Greek counterpart, was *civitas mundi* or practical results. The demise of this empire did not help much. The vacuum of power thus created was promptly filled by the Catholic Church and Byzantium – not quite the prototypical devotees of secular scholarship.

The chief concern of the Catholic Church was *civitas dei*, preparation for the latter world. Its education program consisted of schools which taught Biblical principles and the geometric, musical, and arithmetic compilations of Boethius (480-524 AD). To its credit, it did sanction the translation of ancient works from Arabic and Greek into Latin[1], but it also instilled a fear of knowledge in the common populace. Study of philosophy or mathematics, bastions of creative thinking, were officially condemned. St Augustine, for instance, is accredited with saying

> Quapropter bono christiano, sive mathematici, sive quilibet impie divinantium, maxime dicentes vera, cavendi sunt, ne consortio daemoniorum animam deceptam, pacto quodam societatis irretiant (Da Genesi ad Litteram, II, xvii, 37)[2].

It has been claimed that since the Latin word *mathematici* is derived from the similar Greek word meaning of "something learned", Augustine must have used the word to refer mainly to astrologers[3], although no

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[1] Hence, possibly the existence of some historical accounts that purport to reduce the role of Middle Eastern mathematicians to mere translators of Greek classics.

[2] The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell.

[3] And thus the translation by J.H. Taylor in *Ancient Christian Writers* (1982): Hence, a devout Christian must avoid astrologers and all impious soothsayers, especially when they tell the truth, for fear of leading his soul into error by consorting with demons and entangling himself with the bonds of such association.

credible evidence is provided as to why the word was not used to mean an alchemist, a natural philosopher, or indeed a mathematician. In any case, the moral of the story that even as enlightened a scholar as St. Augustine considered those whose works did not agree with the teachings of the Church to be "consorting with demons," is unaffected by this lexical argument.

As for Byzantium, although Constantinople remained a safe haven for the collected works of ancient Greece until it was overrun by the Turks in 1453, its only achievement in the field of mathematics was preservation of knowledge; no major discoveries or advances were made by Byzantine mathematicians.

Clearly, these circumstances could not have been very conducive to the proliferation of critical thinking or to the advancement of sciences.

On the bright side, universities began to arise from large monastery schools with Dominican and Franciscan monks as teachers. The first was the University of Salerno, near Naples in Italy, founded in the 9th century for the further study of medicine. The late Middle Ages saw the establishment of many other new universities, initially at Bologna and Padua in Italy and then at Oxford and Cambridge in England in the 13th century.

And there were, of course, some independent thinkers though too few to imprint their perspectives upon the general populace. Amongst these scholars one should mention Adelard of Bath[4] (1090-1150), Leonardo of Pisa (Fibonacci) (1175-1250) who introduced Arabic numerals to Europe, and Roger Bacon (1214-1294). Bacon asserted his readiness to demolish the inherited works of Aristotle on the basis that they were erroneous and disseminated ignorance, and proposed to replace Aristotelian authority by

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[4] Adelard of Bath was a 12th century English scholar who traveled widely, especially to Islamic lands. He studied at Tours and taught at Laon. In addition to original work, he translated Greek and Islamic works of astrology, astronomy and mathematics, including the works of Euclid and al-Khwarizmi. His best known work is *Questiones naturalis* (Natural Questions), a work in the form of a Platonic dialogue (with his nephew), where he explored such issues as the shape of the Earth (he believed it to be round), how it remains stationary in space, and how far a rock would fall if a hole were drilled through the earth and a rock dropped in it.

experimentation. Until the publication of *Algorismis Proportionum* (c. 1360) by Nicole Oresme (1323-1382), the Bishop of Lisieux, there were no other significant scholarly achievements.

However, sociopolitical conditions were somewhat different in the Middle Eastern countries (Goldschmidt and Davidson 2005), India, and China (Berggren 2007). The ruling elite were more open to scientific advances. Most of these despotic monarchs and administrators, who were otherwise as corrupt and as ferocious as their European counterparts, respected, supported, and funded scholarly activities. In this paper, we will briefly review the works of some Middle Eastern scholars of the era, namely, *Mohammad Abu'l-Wafa Al-Buzjani*, *Abu Ja'far Mohammad ibn Musa Al-Khwarizmi*, and *Nasir al-Din al-Tusi* to clearly demonstrate this point.

Abu'l-Wafa was born June 10, 940 C.E. in Buzjan in the Khorasan region now in Iran. He was brought up during the Buyid Islamic dynasty in the period between the Arab and Turkish conquests. He was, along with *al-Quhi* and *al-Sijzi*, one of the more distinguished scientists at Caliph Adud ad-Dawlah's court in Baghdad (Berggren 1996). When in 983, Adud ad-Dawlah's son Sharaf ad-Dawlah became the new caliph, he ordered an observatory to be set up in the garden of the palace in Baghdad. Abu'l-Wafa was present at the opening in 988. The instruments in the observatory included a quadrant and a sextant that were designed by him.

Abu'l-Wafa was the author of *Kitab fi ma yahtaj ilayh al-kuttab wa'l-ummal min 'ilm al-hisab* (The Book of What Is Necessary for Scribes and Businessmen from the Science of Arithmetic). The book was arranged in seven parts, each part containing seven chapters. Part one was on ratios, part two on multiplication and division of integers and fractions, part three on measurements of lengths, areas, and volumes, part four on calculating taxes, part five on problems arising from the exchange and sharing of different types of crops, part six on units of money and payments, and part seven on other business topics. Abu'l-Wafa had also compiled a table of sines accurate to 8 places (whereas Ptolemy's was accurate to 3 places), and was the first mathematician to use the tangent function. He died on July 15, 998 in Baghdad.

A map showing the regions and cities mathematicians of the era lived and worked[5]

Less is known about the details of al-Khwarizmi's life. He was born in 780 C.E. in Baghdad. The name al-Khwarizmi suggests that his origins must be from Khwarizm, south of the Aral Sea in Central Asia. He lived under the caliphs Harun-al Rashid and Al-Mamun, both of whom encouraged the arts, established intellectual disciplines, and were patrons of scientific learning. He died about 850 C.E., possibly in Baghdad.

Al-Khwarizmi's best known book was *Hisab al-jabr w'al muqabala* (The Arithmetic of Completion and Balancing), a hand written copy of which is in the collections of the Oxford University Library. He used the

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[5] Encyclopædia Britannica, Inc.

word *al-jabr* (completion) to mean removing negative terms from an equation, and the word *muqabala* (balancing) to mean reducing positive like terms in an equation. Indeed, it is the title of this text that gives us the word *algebra*, although it is only the first part of the book that deals with algebra proper. The other parts of the book deals with problems concerning inheritance, partitions, law suits, trade, measuring land, digging canals, and geometric computations.

In the "algebra" portion of *Hisab al-jabr w'al muqabala*, al-Khwarizmi first introduced natural numbers and then proceeded to the main topic of this section, namely, solving linear and quadratic equations. He solved these equations by reducing them to one of the following standard forms:[6]

1. $x^2 = x$
2. $x^2 = a$
3. $x = a$
4. $x^2 + ax = b$
5. $x^2 + b = ax$
6. $ax + b = x^2$

He then gave algebraic and geometric methods to solve each one of these types. In the later parts of this section, Al-Khwarizmi also showed how the laws of arithmetic extended to "algebra". For example, he gave the correct solution for products of the form $(ax + b)(cx + d)$.

Al-Khwarizmi also wrote a treatise on Hindu-Arabic numerals. The Arabic text is now lost, but the Latin translation still exists. As a matter of fact, it is the title of this Latin translation, *Algoritmi de numero Indorum* (Al-Khwarizmi on The Hindu Art of Reckoning) that has given rise to the word *algorithm*. He also wrote a major work on geography giving the latitudes and longitudes for 2402 localities.

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[6] It must be kept in mind that al-Khwarizmi did not use any mathematical symbols.

2. **Life of Nasir al-Din al-Tusi**

Undoubtedly, the most creative and most resourceful member of this triumvirate was Nasir al-Din al-Tusi (Ataev 1972, Collomb 2006). Since some of the truly monumental manifestations of Middle Eastern mathematics, especially in the realm of geometry, were realized by him, he was, deservedly, referred to as "*the Euclid of the Middle East"* by some historians.[7] Indeed, as we will see later, some of his works in geometry have been the basis of the solution techniques of some problems in modern kinematics.

Nasir al-Din al-Tusi, whose proper name was *Abu Ja'far Muhammad ibn Muhammad ibn al-Hasan Nasir al-Din al-Tusi*, was born on February 18, 1201 in Tus, close to Meshed in Khorasan, now in Iran. He was also known as *Khawaja-yi Tusi* (the learned scholar from Tus), *Khawaja Nasir* (the learned scholar Nasir), and *Al muhaqqiq* (the researcher). Since his father was a jurist in the Twelve Imam school[8] he was educated primarily in a religious institution. However, concurrently, he was tutored by his uncle in topics such as logic and physics. He also received some instruction in jurisprudence, philosophy, mathematics, medicine, and astronomy (Dabashi 1996, 529).

Tusi's life was fashioned by the conflicts created by the Mongol invasion of the Islamic world, and the following atrocious massacres of its peoples. When in 1214 Genghis Khan started advancing westward, Tusi moved to Nishapur, an important center of learning, to complete his education by further studying philosophy (under Farid al-Din Damad), medicine, and mathematics (under Muhammad Hasib) (Nasr 2006). The Mongol offensive hit the area in 1220. Upon the invitation of Ismaili ruler Nasir ad-Din Abd ar-Rahim,

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[7] For example the German mathematician Clavius (Christoph) (1537-1612) did so in his book *Euclidis Elementarum Libri,* which was published in Frankfurt. The copy we have today is the 1654 edition.

[8] The word imam is used to mean an Islamic leader. There are several different subdivisions of the Shiite sect like the Six Imams, the Seven Imams (Alevis), and the Twelve Imams. The Twelve Imam sect believes that the twelfth imam Mehdi will appear at Armageddon.

Tusi joined the service of the Hashishians[9] and went to the castle of Alamut in the Elbruz Mountains. This was supposed to be an impenetrable fort, and hence one of the few safe and peaceful sanctuaries in the area. Here, in 1232, he wrote a book on ethics, *Ahlaq-ı nasiri*, and dedicated it to Nasir ad-Din Abd ar-Rahim.

When, later, he established a close relationship with Caliph Musta'sam, the Abbasid minister, Ibn Al-Kami grew jealous of him, and, fearing he might lose his position, had Tusi jailed. Tusi, however, remained undaunted, and composed most of his mathematical works while in jail.

The castle was eventually destroyed by Hulagu, a grandson of Ghengis Khan, in 1256. Fortunately for Tusi, Hulagu was greatly fascinated by astronomy and mathematics, and Tusi was immediately released from jail and was treated with utmost deference. He lived the remaining eighteen years of his life in relative prosperity under the protection of Hulagu. He died on 26 June 1274 in Kadhimain near Baghdad and, upon his will, he was buried in the *Imam Musa Kazım* mausoleum near Baghdad.

The total number of his works is estimated to be about sixty-four, a clear proof of his versatility and fecundity. These works cover a wide range of topics such as arithmetic, geometry, trigonometry, astronomy, optics, mineralogy, geography, medicine, logic, philosophy, ethics, and music (Street, 1995). Some of the better known ones are

- Kitāb al-Shakl al-qattā' Book on the complete quadrilateral. A five volume summary of trigonometry.
- Tajrid-al-'Aqaid – A major work on Islamic philosophy
- Al-Tadhkirah fi'ilm al-hay'ah – A memoir on the science of astronomy.
- Akhlaq-i-Nasri – A work on ethics.
- Al-Risalah al-Asturlabiyah – A Treatise on astrolabe.
- Zij-i ilkhani (Ilkhanic Tables) – A major astronomical treatise, completed in 1272.

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[9] This group practiced an intellectual and militant form of extremist Shiite. The name means "*those who consume hashish",* because they used hashish to help them gain courage in their struggle against the establishment of Mongol rule. The word *assassin* is derived from the name of this sect.

A Commemorative Stamp Picturing Al-Tusi[10]

He was also an accomplished poet:

> In fact it was common among Persian Islamic philosophers to write few quatrains on the side often in the spirit of some of the poems of Khayyam singing about the impermanence of the world and its transience and similar themes. One needs to only recall the names of Ibn Sina, Suhrawardi, Nasir al-Din Tusi and Mulla Sadra, who wrote poems along with extensive prose works (Nasr 2006, 167)

Among the writers of science of the period, it is hard to come across another one as didactic and as close to modern scientific methodology as Tusi. To his credit, he was also modest enough to emphasize the superiority of the methods of others when that was indeed the case.

3. **Mathematics of Nasir al-Din al-Tusi**

Tusi was a student of *Kamaladdin ibn Yunus* and *Muin-ud-din Salim*, two of the most prominent mathematicians of the period. He also had a very sound knowledge of Greek, and this vast command of language, combined with his mathematical eruditeness, made it possible for him to study and later translate several classics of Greek mathematics and astronomy into Arabic and Persian. He wrote revised Arabic versions of works by *Autolycus*, *Aristarchus*, *Euclid*, *Apollonius*, *Archimedes*, *Theodosius*, *Menelaus*, and *Ptolemy*. His commentary on Ptolemy is especially important, for in this commentary, *Tahrir al Majisti* (Commentary on Almagest), written in 1247, Tusi introduced techniques to make tables of the sine function (Ibadov 1968). This commentary based on thirteen articles was by far the best translation of Almagest and was instrumental in shaping the astronomy was described in the Dictionary of Scientific Biography as

> *... the first in history on trigonometry as an independent branch of pure mathematics and the first in which all six cases for a right angled spherical triangle are set forth*[11].

Here, Tusi studied plane and spherical trigonometry in a systematic manner, and introduced the Law of Sines for plane triangles (Hairetdinova 1976). He also established a method for solving a spherical triangle with three known angles using the *şeklüzillî* (tangents) *rule*. He was the first mathematician to think of geometric transformations to solve spherical triangles with three known angles [*Kitab-ı Şeklül Kutta*, Fifth Article, p. 182]. We will later illustrate this point by an example.

Kitab-ı Şeklül Kutta was arranged in five articles:

Article I. Geometric Ratios and Theorems on Ratios

Article II. Theory of Quadrilaterals

Article III. Plane Trigonometry

Article IV. Spherical Quadrilaterals

Article V. Spherical Trigonometry

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[10] http://en.wikipedia.org/wiki/File:Nasir_al_din_Tusi.jpg

[11] See Appendix I.

Tusi notes that the writing of this book was finished on the 21st day of Cemaziyulevvel (the fifth month of the Arabic calendar) of the year 658 (that is 1261 C.E.), implying that the book was completed in the observatory founded a year earlier. In this book, he formulated the famous law of sines for plane triangles, $a/\sin\alpha = b/\sin\beta = c/\sin\gamma$, which was one of main mathematical contributions offered in this book (Ibadov 1968).

To give a flavor of the book, let us give here, as an example, the sixth problem in Article V of the book (Dilgan 1958):

Problem: Determine the sides of a spherical triangle *ABC* when all the angles are known.

Solution: Extend the sides *AB* and *AC* to *AD* and *AE* such that $AD = B/2$, and $AE = B/2$. Similarly, extend *BA* and *BC* to *BT* and *BK* such that $BT = B/2$, and $BK = B/2$. Now extend *CA* and *CB* to *CF* and *CH* such that $CF = B/2$, and $CH = B/2$ as shown in Figure 1:

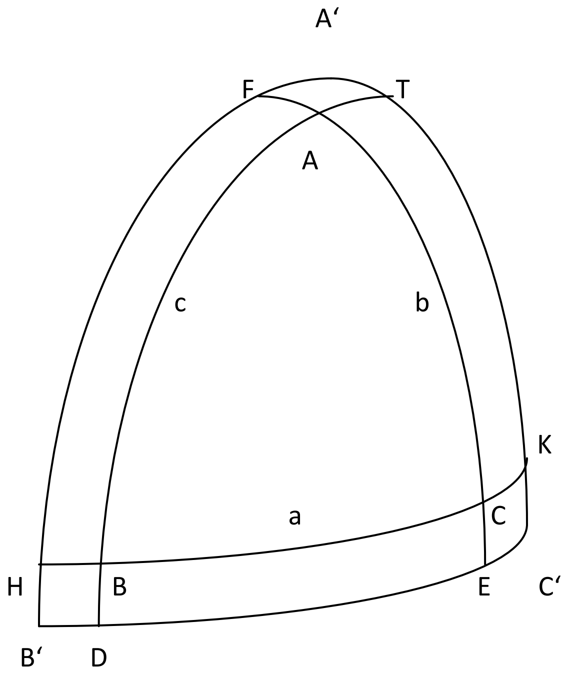

Figure 1

Extend now *DE*, *TK*, and *FH* into great circles that intersect each other at A', B', and C', and thus construct the spherical triangle $A' B' C'$. Obviously, $DE = \angle A$, $KT = \angle B$, and $FH = \angle C$. On the other hand, since $\angle K$ and $\angle H$ are right angles, A' is a pole of *HBCK*. Similarly, B' is a pole of *ECAF*, and C' is a pole of *TABD*. Let the sides of the triangle *ABC* be denoted by *a*, *b*, and *c*, and the sides of the triangle $A' B' C'$ by a', b', and c'. Since by construction

$$A'T + B = C'K + B = \pi/2$$

one gets

$$A'T = C'K = \pi/2 - B$$

implying

$$\begin{aligned} b' &= A'C' \\ &= 2(\pi/2 - B) + B \\ &= \pi - B \end{aligned}$$

Proceeding similarly, one obtains the two other similar identities, consequently

$$A + a' = \pi$$

$$B + b' = \pi$$

$$C + c' = \pi$$

implying that the sides of the triangle $A' B' C'$ are now known. By a previous result (on finding the angles of a spherical triangle when the three sides are known) he had proven, Tusi could now determine the angles $\angle A'$, $\angle B'$, and $\angle C'$ of this triangle. Consequently, he could determine the lengths of the arcs *KH*, *DT*, and *EF*. But by the above construction

$$BC = a$$

$$a + CK = \pi/2$$

$$a + BH = \pi/2$$

implying

$$CK = \pi/2 - a$$

and

$$BH = \pi/2 - a$$

On the other hand, since $HK = A'$, he determined that

$$\begin{aligned} HK &= a + 2\,CK \\ &= a + 2\,(\pi/2 - a) \\ &= \pi - a \end{aligned}$$

implying

$$A' = \pi - \mathrm{a}$$

Proceeding similarly, he obtained the three identities

$$A' + a = \pi$$

$$B' + b = \pi$$

$$C' + c = \pi$$

Thus, Tusi achieved the desired solution by establishing the relations between the elements of two spherical triangles.

As another example of Tusi's geometric adeptness, let us give his proof of the Pythagorean Theorem: Since in Figure 2 below, the areas of triangles GAL and ABC are equal, $LA = BC = a$, and since $\angle GAL = \angle ABC = \angle CAM$, the points L, A, M, and K are collinear. Also note that the area of the parallelogram $AC'DL$, and thus that of the rectangle $CMKD$ is b^2. Similarly, the area of the parallelogram $ABE'L$ and thus that of the rectangle $MBEK$ is c^2. But since the sum of the areas of rectangles $CMKD$ and $MBEK$ is equal to the area of the square $CBED$, we get

$$b^2 + c^2 = a^2$$

as desired.

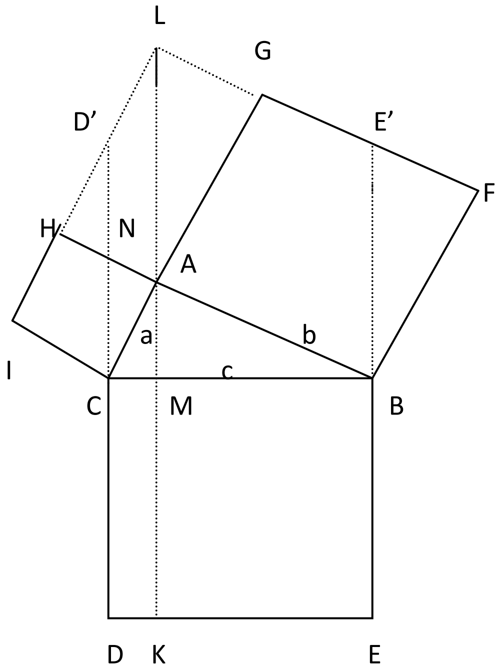

Figure 2

Tusi was astute enough to comprehend that the problems involving spherical triangles and rectangles possessed profound geometrical and physical implications. So it is not surprising that there are some theorems of applied geometry in Kutta that are very similar in nature to those of modern day kinematics. For example, the following is a famous theorem of Tusi:

> *Assume a circle r is rolling without slipping on the inside of a second circle of radius R which is also rolling.*[12] *Assume, moreover, that R = 2r. At the instant when the velocity of the*

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[12] See Appendix II.

> *smaller circle is twice the velocity of the bigger circle, the point of contact of the smaller circle follows the radius of the bigger circle.*

This was the first time that a mathematician was explicitly stating the idea of converting circular motion to linear motion.

Tusi's competence in geometry was perhaps best demonstrated in *Tahrir-î Usul-ül Oklidis* (An Exposé on the Methods of Euclid). This was a commentary on Euclid's *Στοιχεια* (Elements), amended by some additions (Berozashvili 1978). Tusi had two translations of Euclid, one consisting of thirteen, the other of fifteen books. The first one was translated into Latin in 1544 in Rome and published as *Euclidis Elementorum Geometricum Libri.* This translation is kept in Rome along with its Arabic original. The second version is in the Central Library of the Technical University of Istanbul.

One of Tusi's amendments concerned the Fifth Postulate (Rozenfeld 1951, Rozenfeld and Yushkevich 1960, Lévy 1992). Tusi stated this postulate as follows:

> *Let D be a line and let P, Q, R, ... be points on D marked in order in the same direction. Let* D' *be another line coplanar with D. Then, the lengths of the perpendiculars from the points P, Q, R, ... to* D' *either get smaller and smaller or larger and larger.*

Based on this axiom, Tusi proved that the sum of the interior angles of a plane triangle would be 180°, from which he deduced Euclid's Fifth Postulate.

His achievements in number theory were comparable to those of Diophantus. For example, Tusi proved that for any two natural numbers *m* and *n*, the sum

$$(2m+1)^2 + (2n+1)^2$$

could never be a perfect square. The proof was quite simple: Assume the contrary, that is, assume that there is a natural number *k* such that

$$(2m+1)^2 + (2n+1)^2 = k^2$$

Then k^2, and consequently k must be even, that is $k = 2r$ for some natural number r. But then

$$4m^2 + 4m + 4n^2 + 4n + 2 = 4r^2$$

Reducing both sides by 2, one observes that the left-hand side would be odd, whereas the right-hand side would be even. This contradiction proves the result.

In 1256, Tusi also wrote a manuscript on calculating nth roots (Ahmedov 1970). Moreover, he revealed the coefficients of the expansion of a binomial to any power giving the binomial formula and the Pascal triangle relations between binomial coefficients.

Another remarkable mathematical achievement of Tusi was his use of the concept of the derivative, albeit implicitly in the solution of the cubic equation. He considered, for example, the equation

$$x^3 + a = bx$$

where a, b were positive constants.

His argument went as follows: the maximum of $y = bx - x^3$ occurs when $x = \sqrt{(b/3)}$ and $y = 2(b/3)^{3/2}$ (and this is where the implicit use of the derivative comes into play). Thus, the equation $bx - x^3 = a$ has a positive solution if $a \leq 2(b/3)^{3/2}$, that is when

$$b^3/27 - a^2/4 \geq 0$$

this quantity, of course, being the discriminant of the equation (Farès 1995, Hogendijk 1989).

4. **Tusi the Astronomer**

At the beginning of the thirteenth century, astronomy was going through a period of stagnation. More and more, the astronomical tables of the past were proving to be incomplete and even incorrect. As a matter of fact, the only reliable table of the time was *Zic-î Hakimî,* written by Ibn Yunus (950 AD-1008 AD) in Cairo. Tusi knew that better tables could only be constructed in a modern and well-equipped observatory. Luckily, the only person who was powerful enough to order the building of a new observatory was Hulagu who respected and trusted Tusi very much. Moreover, Hulagu believed strongly in astrology, which he equated with observing the positions of celestial bodies. So, when Tusi presented Hulagu with plans for the construction of an observatory, the latter agreed. The building of the observatory began in 1259 and ended in 1262. It was located west of Maragheg, in Azerbaijan, the city that Hulagu had made his capital[13] (Godard 1934 and 1936).

Tusi was made the director of this observatory, and here he produced several important astronomical papers (Kennedy 1984). He invited several well known scientists such as *Muey-yid-ud-Din Urzî* from Damascus, *Fahr-ud-Din Maragî* from Mousul, *Fahr-ud-Din Ahlatî* from Tiblisi, *Necm-ud-Din Douberyanî* from Kazvin, *Muhay-ed-Din Magribî* from Aleppo, and *Fao-Mun Chi* (who had earned the nickname *Sing Sing* (the sagacious one) from China. He designed most of the instruments in this observatory including the azimuth quadrant. The observatory also had a four-meter wall quadrant, instruments to measure one-minute arcs and the radius of the moon, graphometers to observe the angular distance between two stars, astrolabes, instruments that modeled epicycle systems, and gnomons. Among other things, Tusi calculated the altitudes of the sun during the solstice and equinox with a high degree of accuracy. Through a slit at the top of the building, the sunrays were projected vertically upon a surface, and thus, the external motion of the sun was analyzed (Usmanov, 1978). The observatory was also a center for learning. According to the historian, Ibn Shakir it had a very rich library - some 400,000 volumes of books, treatises, and pamphlets.

Tusi had asked Hulagu for a period of thirty years to correct the old Greek and Arab observations and to formulate astronomic tables based on these corrections at the observatory. This period was based on the period of the revolution of Saturn around the sun; for, Tusi had intended to observe planetary motions and the geometric laws governing these motions within the existing bounds of the solar system of the period. When Hulagu found this period excessive, this interval of observations was reduced to thirteen years, the

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[13] The exact coordinates of the observatory were 37E 18N parallel and 46E 15N meridian.

period of the revolution of Jupiter.[14] Yet, ironically, when the results were published as *Zîcî Ilhanî* in 1269, Hulagu was already dead.

Zîcî Ilhanî (The Ilkhanic Tables) was first written in Persian and later translated into Arabic. It contained tables for computing positions of the planets and a catalog of stars. It was based on Tusi's observations and also on works of astronomers such as Hipparkus, Ptolemy, Al-Battanî and Ibn Yunus. Its opening sentences might give us an idea about what scientists of the time had to go through to get their works published:

> *Whence Hulagu Khan conquered the land of the Ismailis, he took me, one of his most helpless servants, out of Tus, which was now part of his country, and commanded me to prepare a catalog of stars.*

The Zic was organized in four parts:

1. Chinese, Greek, Arab, and Persian calendars and conversions
2. Motions of the planets
3. Ephemersis: Tabular statement of the assigned places of a celestial body for regular intervals
4. Various astronomical problems

It was based on this work that about three and a half centuries later in 1627, Kepler was able to publish his astronomical tables *Tabulae Rudolphinae*.[15] His model for the planetary system is believed to be

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[14] Tusi writes in Zîcî Ilhanî
The masters of the science of astronomy had notified us that a catalog of stars could be completed after at least a thirty-year period of observations which corresponds to the conjunction time of the seven planets [sun, moon, Venus, Jupiter, Mars, Mercury, and Saturn. Keep in mind the system was the geocentric system.] *Indeed, a period longer than thirty years would be more suitable... but we were ordered to complete the observations in a shorter time, at most within twelve years. The astronomers of Maragheg had to abide by this wish.*

[15] A copy of Zîcî Ilhanî, written by Tusi's son, Asîl-ad-Din, is in Bibl. Nat. de Paris (Blochet, E. ***Catalogue des Manuscrits Persians***, vol. II, No. 779). The inscription at the beginning reads

> *This table is the own handwriting of the son of Khawja Nasir-ad-Din Asîl-ad-Din. May he rest in peace.*

which shows that this is possibly the only such authentic copy. Another handwritten copy is in Istanbul at the Library of the most advanced of his time, and was used extensively until the development of the heliocentric model in the time of Nicolaus Copernicus. Between Ptolemy and Copernicus, he is considered by many to be one of the most eminent astronomers.

Other astronomical works include *Tahrir-ul-Mijistî* (Mamedov 1963, Dovlatova 1969, Abdulkasumova 1977) and *Al-Tadhkira fi'ilm al-hay'a* (Memoir on Astronomy). The latter is also in four parts (Livingstone 1973):

1. An introductory part covering geometry and kinematics. This section deals with objects at rest and subject to simple and complex motions.
2. General Astronomy: Equinoxes, a discussion of Ibn Heysem's cosmogony theory, the phases of the moon (Hartner 1969), a criticism of Ptolemy's theories of epicycles and a proposal to correct them.
3. General geodesy: The effects of the Earth on celestial bodies.
4. Dimensions of and distances between various planets.

According to *Dictionary of Scientific Biography*, in this book, Tusi

> *...devised a new model of lunar motion, essentially different from Ptolemy's.*

and

> ... [Tusi] *for the first time in history of astronomy, employed a theorem invented by himself which, 250 years later occurred again in Copernicus, "De Revolutionibus".*

The theorem referred to is the well-known *Tusi-couple,* which resolves linear motion into the sum of two circular motions, thus removing all parts of Ptolemy's system not based on uniform circular motion (Hartner 1969, Gingerich 1984, Kennedy 1984, Ragep 2001a and 2001b).

Tusi also stated a version of the law of conservation of mass. He wrote that a body of matter is able to change, but is not able to disappear (Alakbarov 2001)

Kandilli Observatory.

> A body of matter cannot disappear completely. It only changes its form, condition, composition, colour and other properties and turns into a different complex or elementary matter.

6. **Conclusion**

We conclude by remarking that the measurement of advance in sciences should be a based on our collective successes as a human race, and not by what goes on in one continent alone. "Darkness" and "enlightenment" will dominate one culture or another at different times. Intellectual conquests over institutions which invest heavily on the preservation of status quo cannot be properly explored if they are detached from the political, social, and economic conditions of the times and the regions.

Mathematics, as any other branch of learning, is a human activity and hence is intensely culture-based and is certainly dependent on the socio-political conditions of the period. That is why it has been developed by different cultural groups at different times: fundamentals that started in Egypt and Greece, were then carried further by Middle Eastern mathematicians, and following Modernism, were abstracted and perfected by European scholars, who enlarged our domain of knowledge by discoveries of their own. All these different stages of development are interrelated and interdependent - mathematics is truly a global process. Understanding this cyclical and universal nature of our discipline should help us appreciate it as cultural phenomenon without borders and as one that never had a "dark" period.

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Appendix I

Spherical Trigonometry

Let *S* be a unit sphere. A plane intersects the surface of *S* in a *great circle* if the plane passes through the center *O* of the sphere. Let *A*, *B*, and *C* be three distinct points on the sphere, not all on the same great circle. The set of points *O A B, O A C,* and *O BC*, each determine a unique great circle. The *spherical triangle* formed by taking, in each case, the shorter of the two possible arcs of the great circle is called an *Euler triangle*.

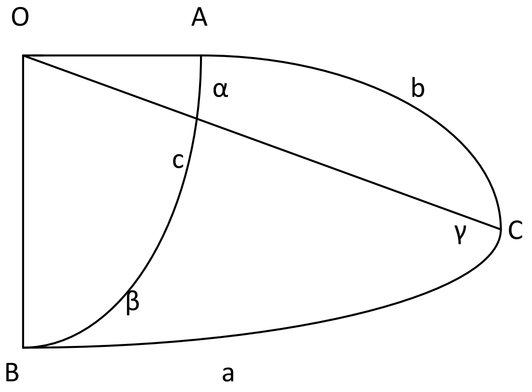

Figure 3

Since the radius *r* of the sphere is 1, by virtue of the formula $s = r\vartheta$, the sides of the triangle are given by the angles *BOC*, *AOC*, and *AOB*, respectively. Each side and each angle in an Euler triangle is less than π. The sum of the three interior angles is always greater than π, and the quantity

$$P = \alpha + \beta + \gamma - \pi$$

is called the *spherical excess*. The area of a spherical triangle on a sphere of radius *r* is given by

$$P\, r^2 .$$

In the solution of spherical triangles, there are two cases:

I. Right Spherical Triangles

Let us assume that the right angle is γ. Given any two parts of the triangle, the remaining three parts can be found by *Napier's Rules*:

$\sin(a) = \sin(\alpha)\sin(c) = \tan(b)\cot(\beta)$

$\sin(b) = \sin(\beta)\sin(c) = \tan(a)\cot(\alpha)$

$\cos(\alpha) = \cos(a)\sin(\beta) = \tan(b)\cot(c)$

$\cos(c) = \cos(a)\cos(b) = \cot(\alpha)\cot(\beta)$

$\cos(\beta) = \sin(\alpha)\cos(b) = \tan(a)\cot(c)$

II. Oblique Spherical Triangles

We must distinguish six possible cases:

1. If the three sides are given, set $s = (a + b + c) / 2$, and

$$d^2 = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin(s)}$$

Then,

$$\tan\left(\frac{\alpha}{2}\right) = \frac{d}{\sin(s-a)}$$

with corresponding formulas for β and γ.

2. If the three angles are given, set $S = (\alpha + \beta + \gamma) / 2$, and let

$$D^2 = \frac{-\cos(S)}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}$$

Then,

$$\tan\left(\frac{a}{2}\right) = D\cos(S-\alpha)$$

with corresponding formulas for *b* and *c*.

3. Given two sides and the included angle, we use the formulas

$$\tan[(\alpha - \beta)/2] = \cos(\gamma/2)\sin[(a-b)/2]/\sin[(a+b)/2]$$

and

$$\tan[(\alpha + \beta)/2] = \cot(\gamma/2)\cos[(a-b)/2]/\cos[(a+b)/2]$$

to get the sum and the difference of, and thus, the two angles. The third side can be found from

$$\tan(c/2) = \tan[(a-b)/2]\sin[(\alpha+\beta)/2]/\sin[(\alpha-\beta)/2]$$

or from

$$\tan(c/2) = \tan[(a+b)/2]\cos[(\alpha+\beta)/2]/\cos[(\alpha-\beta)/2]$$

These formulas are known as *Napier's analogies*.

All other cases can be solved by Napier's analogies and the *Law of Sines*:

$\sin(a)/\sin(\alpha) = \sin(b)/\sin(\beta) = \sin(c)/\sin(\gamma)$

Appendix II

Cycloids

Let a circle of radius *r* roll along the *x*-axis without slipping. A fixed point *P* on the circumference of the circle will describe a path called a *cycloid*, whose parametric equations are

$$x = rt - r\sin(t)$$

$$y = r - r\cos(t)$$

If the point P is, instead, fixed on the radius of the circle, then the path obtained is called a *curtate cycloid* (*trochoid*) with parametric equations

$$x = rt - q\sin(t)$$

$$y = r - q\cos(t)$$

where $q < r$ is the distance between P and the center of the circle. In case $q > r$, the path obtained is called a *prolate cycloid* (*trochoid*).

Let now a point P be fixed on the circumference of a circle with radius r, and let this circle roll without slipping on the circumference of another circle with radius R. If the first circle rolls on the outside of the second circle, the path obtained is called an *epicycloid.* Its parametric equations are

$$x = \mathrm{Ar}\cos(\vartheta) - r\cos(A\vartheta)$$

$$y = \mathrm{Ar}\sin(\vartheta) - r\sin(A\vartheta)$$

where $A = (R + r) / r$, and ϑ measures the angle between the positive x-axis and the line joining the centers of the circles.

If the first circle rolls on the inside of the second circle, the path obtained is called a *hypocycloid*. Its parametric equations are

$$x = Br\cos(\vartheta) + r\cos(B\vartheta)$$

$$y = Br\sin(\vartheta) - r\sin(B\vartheta)$$

where $B = (R - r) / r$, and ϑ measures the angle between the positive x-axis and the line joining the centers of the circles.