Al-Maghribî’s Mecca Problem Meets Sudoku

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Abstract

In this paper we will talk about an old and rather obscure puzzle known as the Mecca Problem, discuss some of its interesting generalizations, and see how it relates to the modern day Sudoku and indeterminate systems.
The Statement and the Solution of the Original Problem

What does an old riddle of a seventeenth century mathematician, Ali bin Veli Ibn Hamza al-Cezâirî (? – 1614), commonly known as Ibn Hamza Al-Maghribî, have to do with Sudoku, a modern-day popular Japanese puzzle that is a staple of our daily newspapers?
The Statement and the Solution of the Original Problem

Ibn Hamza was born in Algiers – in fact, the title Al Maghribî (the one from the west) seems to be given to him because of his Algerian origins. Towards the end of the sixteenth century he moved to Istanbul, then the capital of the Ottoman Empire, to complete his education and most likely stayed there until his death.
The Statement and the Solution of the Original Problem

He was an algebraist who also made important contributions to probability, hydrodynamics, mechanics, medicine, and geology.

The only written work he left behind was a treatise called *Tuḥfetū'l - Âdâd lizevil Rüşd ve's – Sedad*. (*The Ornament of Numbers*).
The Statement and the Solution of the Original Problem

This was a five hundred-page book written in Ottoman Turkish containing some elementary theorems of Euclid and solutions of various Diophantine problems, and even some form of logarithms (Djebbar, 2003). It comprised an introduction, four sections and an appendix.
The Statement and the Solution of the Original Problem

The first section was on the properties of integers and the four basic operations on integers. The second section was devoted to developing rules of computations with rational and irrational numbers and methods of finding square, cube, and fourth roots.
The Statement and the Solution of the Original Problem

The third section dealt with approximate solutions of equations by various techniques such as the *method of proportions* or the *regula falsi* method. The fourth section presented some theorems of elementary geometry, and some formulas to calculate the areas of triangles, rectangles, circles, and volumes of regular solids.
The Statement and the Solution of the Original Problem

In this section, Ibn Hamza also described some algebraic manipulations such as *al-jabr* (completion - that is, removing negative like terms from an equation by adding equal positive quantities to both sides of the equation), and *muqabala* (balancing - that is, reducing positive like terms in an equation by subtracting equal positive quantities from both sides of the equation.) The ideas were, in all likelihood, borrowed from *al-Khwarizmi* (780 - circa 850), who had introduced them a few centuries earlier in his book, *Hisab al-jabr w'âl muqabala*. 
The Statement and the Solution of the Original Problem

The appendix - more or less the most original part of the book - contained several fascinating problems, including the one we are about to discuss, that were solved by the author by some peculiar methods.
The problem we are interested in can be stated as follows: A landowner who has 81 trees, gets every year 1 unit of fruit from the first tree, 2 units from the second tree, ..., and 81 units from the eighty-first tree. How should he divide these trees among 9 inheritors so that each inheritor gets 9 trees and an equal amount of yearly produce?
The Statement and the Solution of the Original Problem

In his book, Ibn Hamza referred to this problem as the *Mecca problem*, possibly because it was suggested to him by an amateur Indian mathematician, Molla Mohammed, on the way to Mecca circa 1590 (Dilgan 1957).
The Statement and the Solution of the Original Problem

Since the 81 trees will yield a total of
\[ 1 + \ldots + 81 = \frac{(81 \cdot 82)}{2} = 3321 \]
units per year, each inheritor should get \( 3321 \div 9 = 369 \) units of fruit per year. Thus, the problem is equivalent to solving a \( 9 \times 81 \) indeterminate system of equations

\[ x_{11} + x_{12} + \ldots + x_{19} = 369 \]

\[ x_{21} + x_{22} + \ldots + x_{29} = 369 \]

\[ \vdots \]

\[ x_{91} + x_{92} + \ldots + x_{99} = 369 \]
The Statement and the Solution of the Original Problem

Here $x_{ij}$ stands for the $j^{th}$ tree inherited by the $i^{th}$ heir, subject to the conditions that $x_{ij} \neq x_{i'j'}$ for $i \neq i'$ or $j \neq j'$. 
The Statement and the Solution of the Original Problem

Here is Al-Maghribî’s rather elegant original solution (Dilgan 1957): Number the trees 1, ... , 81 according to the amount they yield, and place these numbers in a table in the following way: Put 1 in the (1,1) position, then continue to the right along the first row by placing the numbers 2, 3, ..., 9 in consecutive cells. Now move to cell (2,2) and put the next number, 10, in this position.
The Statement and the Solution of the Original Problem

Continue to the right along the second row by placing the numbers 11, ..., 17 in the rest of the cells. Place the next integer, 18, in cell (2,1). Then start at cell (3,3) with the integer 19 and continue to the right by placing successive integers to fill up the remaining cells in the third row. Fill in the first two positions of this row by integers 26 and 27. Continue in this fashion to obtain the following table (Table 1):
The Statement and the Solution of the Original Problem

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The Statement and the Solution of the Original Problem

Note that the sum of each column is equal to 369. Thus, a solution to the problem is that the first inheritor gets trees numbered 1, 18, 26, 34, ..., 74; the second inheritor gets the trees numbered 2, 10, ..., 75; ...; and the ninth inheritor gets the trees numbered 9, 17, 25, ..., 73.
The Statement and the Solution of the Original Problem

We notice that along with the column sums being 369, the secondary diagonal sum is also 369. However, this table is not a magic square: the numbers in the rows (except for the fifth row) and along the main diagonal do not add up to 369. Strictly speaking (see footnote 2), it is not a Latin square either, for although we have an $n \times n$ matrix where no entry occurs twice in any row or column, these entries do not satisfy the condition $1 \leq x_{ij} \leq n$. 


The Statement and the Solution of the Original Problem

There is a simpler way of perceiving the table representing Al Maghribî’s solution. Let us subtract 9 from each entry of the second row, 18 from each entry of the third row, and so on, so that we subtract 72 from each entry of the last row, that is let us construct a table with entries $y_{ij}$ defined by the formula

$$y_{ij} = x_{ij} - 9(i - 1)$$

for $2 \leq i \leq 9$, and all $j$, as depicted below (Table 2):
The Statement and the Solution of the Original Problem

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</table>
The Statement and the Solution of the Original Problem

We note that Table 2 is nine rows of permutations of the first nine integers. Let us also remark that any finished Sudoku puzzle satisfies this general pattern as well. Moreover, if we start with any completed Sudoku puzzle and add the sequence 0, 9, ..., 72 to this array, we would obtain a different solution to Al-Maghribî’s problem.

In fact, after this transformation, the resulting table is a Latin square and the results of next section can also be proved by taking advantage of this structure. This also gives us a multitude of solutions to the original problem, namely 5524751496156892842531225600 solutions – the number of different $9 \times 9$ Latin squares (Bammel and Rothstein 1975).
Generalization of the Problem

In this section we will consider a generalization of the problem. To maintain the historic flavor of the topic, our generalization will be structured after Al-Maghribî’s original solution. A simpler version of this generalization will be given in Section 3.
Generalization of the Problem

Let $i, j, k, m, n,$ and $p$ denote positive integers. Suppose we have $m = n^2 > 1$ trees, such that each year one gets 1 unit of produce from the first tree, 2 units from the second tree, ..., and $m$ units from the $m^{th}$ tree. How should these trees be divided among $n$ inheritors so that each inheritor gets $n$ trees and an equal amount of yearly produce?
Generalization of the Problem

Equipped with our experience of the solution of the special case $n = 9$, let us establish an $n \times n$ matrix $x_{j,k}$ such that

$$x_{j,k} = (j - 1) n + (k - j +1) \quad \text{if } j \leq k$$

$$x_{j,k} = jn + (k + 1 - j) \quad \text{if } j > k$$

as depicted in the following table (Table 3):
## Generalization of the Problem

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Generalization of the Problem

We will now study some interesting properties of Table 3. The first property shows us that this table has, in fact, some of the characteristics of an odd magic square starting with 1.

Property 1. Let \( \sigma_n = \frac{1}{2} n (n^2 + 1) \). Then, all column sums are equal to \( \sigma_n \).
Generalization of the Problem

Proof. Let the sum of the numbers in the $k^{th}$ column be denoted by $S_k$. Then

$$S_1 = 1 + 2n + (3n - 1) + (4n - 2) + ... + [jn - (j - 2)] + ... + [n \cdot n - (n - 2)]$$

$$= 1 + n \frac{(n^2 + n - 2)}{2} - \frac{(n^2 - 3n + 2)}{2}$$

$$= n \frac{(n^2 + 1)}{2}$$

Now by the definition of the $x_{j,k}$'s,

$$x_{j,2} = x_{j,1} + 1$$
except for the diagonal position where

\[ x_{2,2} = x_{2,1} - (n-1) \]

Thus,

\[ S_2 = S_1 + (n-1) - (n -1) \]

\[ = S_1 \]

Proceeding inductively,

\[ S_k = S_{k-1} + (n - 1) - (n - 1) \]

\[ = S_{k-1} \]

for all \( k = 3, \ldots, n \), proving the theorem.
Generalization of the Problem

Property 2. If $n$ is odd, the sum of the entries along the secondary diagonal is $\sigma_n$.

Proof. If $n$ is odd, by formula (1), for $k = 1, \ldots, (n-1)/2$, the elements in the secondary diagonal corresponding to rows $k$ and $(n - k + 1)$ add up to $n^2 + 2$. Adding all of these we get

$$\left[ \frac{(n - 1)}{2} \right] \left[ n^2 + 2 \right]$$
Generalization of the Problem

The only element along the secondary diagonal that is unaccounted for in this sum is \( x_{(n+1)/2,(n+1)/2} \) which by formula (1) is equal to

\[
[(n+1)/2 - 1] \cdot n + 1 = (n^2 - n + 2) / 2
\]

Thus, the sum of the elements along the secondary diagonal is

\[
[(n - 1)/2] \cdot [n^2 + 2] + (n^2 - n + 2) / 2 = \sigma_n
\]
Generalization of the Problem

Property 2 does not hold when $n$ is even. A simple counterexample is provided by the following $4\times4$ matrix (Table 4)
## Generalization of the Problem

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Generalization of the Problem

Here, $\sigma_4 = 34$, whereas the secondary diagonal sum is 36.

However, even in the case $n$ is even, something can be said about the secondary diagonal sum:

**Property 3.** If $n$ is even, the sum of the elements along the secondary diagonal is $\sigma_n + n/2$. 
Generalization of the Problem

Proof. If $n$ is even, the elements on the subdiagonal corresponding to rows $k$ and $(n - k)$, for $k = 1, \ldots, n/2$ all add up to $(n^2 + 2)$; hence, the sum of these numbers will be

$$(n/2) (n^2 + 2) = \sigma_n + n/2$$
Generalization of the Problem

As we remarked earlier, the elements along the main diagonal do not add up to $\sigma_n$:

**Property 4.** For any $p \geq 1$ let $T_p = \frac{p(p + 1)}{2}$. Then, the sum of the elements along the main diagonal is $\sigma_n - T_{n-1}$, that is, the main diagonal sum can never be $\sigma_n$. 
Generalization of the Problem

Proof. In case \( n \) is odd, the elements on the main diagonal corresponding to rows \( k \) and \((n-k+1)\) for \( k = 1, \ldots, (n - 1)/2\), add up to \( n^2 - n + 2 \). Summing, we get

\[
\left\lceil \frac{n - 1}{2} \right\rceil \frac{n^2 - n + 2}
\]

This sum accounts for every element along the main diagonal except for \( x_{(n+1)/2,(n+1)/2} \). Since

\[
x_{(n+1)/2,(n+1)/2} = \frac{n^2 - n + 2}{2}
\]

the sum of all the numbers along the main diagonal will be

\[
\left\lceil \frac{n - 1}{2} \right\rceil \frac{n^2 - n + 2} + \frac{n^2 - n + 2}{2}
\]

\[
= \sigma_n - T_{n-1}
\]

In case \( n \) is even, a similar argument shows that we have \( n/2 \) elements adding up to \( (n^2 - n + 2)/2 \), proving the result.
Generalization of the Problem

So far, we did not say anything about the row sums except to remark that, in general, they are not equal to $\sigma_n$. We remedy that situation by the following results:
Generalization of the Problem

Proposition 1. Let $j$, $k$ be any two integers satisfying $1 \leq k \leq n - 1$, and $1 \leq j \leq n - 1$. Let $x_{j,k}$ denote the entry in the table in the $j^{th}$ row and $k^{th}$ column. Then,

If $j \neq k$, $x_{j+1,k} - x_{j,k} = n - 1$
If $j = k$, $x_{j+1,k} - x_{j,k} = 2n - 1$

$x_{j+1,n} - x_{j,n} = n - 1$ for all $j$.

Proof. Follows from the formula (1) by straightforward subtraction.
Generalization of the Problem

**Property 5.** The sum of the elements along the $k^{th}$ row is $[(2k - 1) n^2 + n] / 2$.

**Proof.** Since, by construction, the smallest number in the $k^{th}$ row is $(k - 1)n + 1$, and the largest one is $kn$, and since each number in between exists once and only once, by rearranging these in ascending order, we see that the elements in the $k^{th}$ row are

$$(k - 1) n + 1, (k - 1) n + 2, ..., (k - 1) n + n$$

Summing, we get

$$(k - 1) n^2 + T_n = [(2k - 1) n^2 + n] / 2$$
Generalization of the Problem

Property 5 can be used to show that the fact that the sum of the elements in the fifth row of Table 1 equaled the column sum was not a mere coincidence:

**Corollary 1.** *For each odd integer n, there exists a row, namely, the (n+1)/2 row, such that the row sum is σₙ.*

**Proof.** Solving the equation

\[
\frac{(2k - 1) n^2 + n}{2} = \frac{n (n^2 + 1)}{2}
\]

for k, we get

\[
k = \frac{(n + 1)}{2}
\]
Remarks

None of the tables 1, 2, 3, or 4 need start with 1. If we start with any natural number $r$, all the above results (with obvious translations) would remain the same.
Remarks

Here is simpler generalization of the problem. Again, Suppose $n^2 > 1$ trees to be divided among $n$ inheritors, where tree $n$ yields $n$ units of produce per year. Let us start out by writing $n$ mutual derangements of the first $n$ integers (Table 5).
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</tbody>
</table>
Now, let us add the sequence $0, n, 2n, 3n, \ldots, (n - 1)n$ to the various rows. Each column of table so generated from Table 5 has common column sums, and the trees designated for inheritor $j$ are the trees with labels in column $j$. 
The generalized problem is equivalent to finding a solution of the following \( n \times n^2 \) indeterminate system of equations:

\[
\begin{align*}
    x_{11} + x_{12} + \ldots + x_{1n} &= \sigma_n \\
    x_{21} + x_{22} + \ldots + x_{2n} &= \sigma_n \\
    \vdots & \\
    x_{n1} + x_{n2} + \ldots + x_{nn} &= \sigma_n
\end{align*}
\]
A Variation

Here is a variation a la Sudoku on his puzzle proposed by Andrew Simoson of King College: The landowner labeled his trees 1 to 81 according to their fruitfulness. On a sandy region of his orchard, he drew a $9 \times 9$ grid. After much trial and error he succeeded in entering all 81 integers into the grid so that the column sums were all the same. Thus, the first son would receive the trees labeled as in the first column, the second son the trees labeled as in the second column, and so on.
A Variation

As he was admiring his solution and before he could write it on some valuable paper, an infrequent rain rendered some cell numbers illegible. He recovered the first row easily enough. But what about the empty cells in the grid below. Can you help him recover his solution?
### A Variation

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REFERENCES

