A few misconceptions in mathematics

... according to Lucian M. Ionescu ...

History & Pedagogy of Mathematics
(ISU conference - April 13, 2014)
Abstract

The History of Mathematics provides examples of mathematicians’ life, work, motivation and how the theory evolved etc. We tend to accept most developments, teaching them almost in “as is” condition, without reassessing theories from a “big picture”, up-to-date, perspective.

I will focus on the role of feedback in math, and argue, via some examples, for the need to break away from the historical developments, rethink old paradigms and teach the new math first etc; this is the current practice in most other areas of R&D (sciences, industry, tech, software, etc.)
Lessons from History (-)

Keeping Math and its history together is “good”:

1) We see math is not God given; it’s man produced: artisanal / craftsmanship (author dependent);

2) “Errare humanum est” (errors ...);

3) We disregard 20\textsuperscript{th} century as “history” (use log t :)

4) Implementation & Interface are modern terms, NOT associated with math theories ;(  

The lessons I like to emphasize (+)

1) Great mathematicians were helped by teachers 1-on-1: advice / “shortcut” provided etc.
2) Their lives, while passionate for math, were (almost always) difficult (like most artists’ lives);
3) Breakthrough theories were usually 1st opposed
4) Carrying with you a Number Theory book helped!
What are we doing & why?

- **Poincare**: having fun & modeling nature’s beauty

- We are **leading** the new generations on a historically chosen path … so we miss some “new opportunities” and especially “new” insights! Therefore: we “lead” while we “miss”, so:

  “lead” + “miss” = “miss” + “lead” = “Mislead”

QED

(what, I missed an “s” ? … :)
FEEDBACK in Mathematics

- FEEDBACK <-> Control (Cybernetics) / LIFE etc.
- Once we understand things better, we should come back and “optimize” the Math-Code and improve the Math-Interfaces. Is it useful rewriting mathematics? In computer Science: NO! not enough time! But when teaching 300 years old theories ... plenty of time!
- In mathematics, concepts & theories are much more stable & “global”; even too permanent, final!
Example: the norm

- The concept of **norm**: analysis norm (subadditive) vs. algebraic norm (↔ quadratic form);

- **Gouvea**: “both terms have been **standard** for such a long time that there is no chance of EVER changing them.” (my emphasis).

- That’s a harmless ambiguity (context dependent; algebra & analysis never mix!); there are other worth addressing and renaming, changing, do something about them!!
Algebra: it’s “axiomatic”, a methodology

- Algebra “is” math taught the axiomatic way:
  1) The standard approach: groups, rings, fields
  2) Groups: Abelian or Non-Abelian etc.

- Better: teach math concepts as they arise from examples & grouped by meaning / relations with applications … (compare with teaching geometry axiomatically first! ha,ha - no way, right?)
Abelian GROUPS are **Spaces of Vectors**

- Is the best (K-5) playground for number theory, algebra and geometry: $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

- What is $\mathbb{Z}_{15}$, really? It is ...

  1) A torus: $\mathbb{Z}_3 \times \mathbb{Z}_5$ (picture with huge $p$ & $q$)
  2) Adding & scaling vectors: it’s **discrete** linear algebra …
  3) A discrete space with coordinates: it’s also **CALCULUS**! (same concepts).
What can you do with a doughnut?

CALCULUS on $\mathbb{Z}_3 \times \mathbb{Z}_5$:
1) Lines, slopes etc.
2) Congruence arithmetic for free …
3) From finite differences, sums & FTC on $\mathbb{Z}$:
   \[ DF(n) = F(n+1) - F(n), \quad \text{Int}_a^b f(k) \, dk = \text{Sum}_a^b f(k) \]
to partial derivatives on $\mathbb{Z}_{15}$, vector calculus and what not! ($\& Y_0$!)
Finite FIELDS are Finite Geometries

- “Fields” is not a reserved keyword: slopes, vectors & strawberry fields forever ...
- FF are “Number Systems” (+ & .) and Geometries a la Klein (ex. $\mathbb{Z}_5$):
  \[ L: (F_5^x, .) \rightarrow \text{Aut}(\mathbb{Z}_5,+) \] … lines $L_m(x) = mx$.
- Multiplication by 3 in $\mathbb{Z}_5$ (the “finite field”) is a change of scales (homothety) etc.

[Greens’ Geometry <-> Descartes’ Analytic G., Alg. NT & 

Time is Change: Discrete (Action=\h)

- Zeno was right … There is a minimal action: Planck’s constant … (Ancient Greeks are amazing!)
- (“FINITE FIELDS”) Think of “discrete time” as \( \mathbb{Z} \) acting on a space, a la Markov Chains (it’s FINITE MATH. for lucky students), or K-5 games of wit on a ring o’ roses (is a regular polygon better?), pockets full of problems of congruences and paths / walks (\( 5 \text{ y/o calculus} \)), and we all sit down (& solve them :)
REAL Numbers are a figmentation ...

- We teach the “real line” and continuity: there are no “real” such things ... everything is discrete! (Greeks knew it; we are gradually finding it out)
- The “completion” of the rationals is a historical mistake / over-reaction!
- Liouville’s constant: $\sum_{n=1}^{\infty} 10^{-n!}=0.11000100…$ is NOT a “number”: doesn’t “measure” anything; it’s a process ...
Teach (about) Designing Mathematics

- Explain (story-like) the process of extending number systems: N -> Z -> Q and -> R, as decimal numbers: “just in case we need more” ..
- Teach more modern mathematics: symmetries, translations & rotations, tied to other areas AND applications (crystals, Platonic solids, graphs etc.)
- Anything AWAY from “continuum” will do :)

Recycle “old” concepts (Cont. -> Disc.)

Most concepts, results and methods which were developed in the context of real numbers, work and are appropriate in the discrete case; from derivative and integrals, to harmonic functions and Riemann Surfaces ... Scratch “most”: ANYTHING can be “imported” in Discrete Math!
“Republicans or Democrats”?

- Well ... variation, diversity, etc. are needed; at least alternation! (it’s “Ok”; a harmonious balance of opposites is much better ... instead of a battle ... :)

- Trends in Mathematics: from Geometry (Greeks) to Algebra (Descartes etc.) => “Abstract Nonsense” (category theory per se, schemes & topoi etc.) ...

- It’s like from polyphonic to dodecaphonic math; we need ... (drum roll) ...
Homecoming: any time soon?

- Mathematics is the **Language of Sciences**, modeling reality ... and the queen does some abstract art too, for fun :)

- Like “flux & reflux”, we need to come back to Discrete, Concrete & Finite Mathematics:
  1) **applications** (chemistry, physics, biology etc)
  
  2) zeta functions / graphs ([Audrey Terras](https://www.audreyterrascollections.com/))

(a picture is a 1000 words & silence is golden ... sometimes :) why flwrs
(Some) Conclusions

1) Don’t hesitate to “rewrite history”! Ex: talk about discrete vector spaces $\mathbb{Z}_n$ pictorially & intuitively;

2) Use abstract algebra keywords; ex. “math objects” and “morphisms” (no definitions: JUST EXAMPLES! teach when to use these terms ...)

3) Limit the rigour in mathematics: unleash the bird, enough raising frogs :) (see “Birds and Frogs” by Freeman Dyson)
... and The 5 Commandments

Thou shalt have Math:

1) **Algebraic**: focus on structures (of natural objects)
2) **Intuitive**: concerned with geometry ($\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$)
3) **Dynamic**: comparing objects (time=change)
4) **User Friendly**: focus on interface (=> no losers :)
5) **Iconic**: concepts <-> pictures (search Google :)
and facts / theorems <-> diagrams (proofs to go ... :).
More concretely ...

- Phase out the “real line” as a central topic, use the integer line, and finite number systems (finite circles / regular polygons), rationals as slopes in ZxZ etc.

- Tilt the balance from Numbers to Geometry & Algebraic Structures (objects); functions are “properties”, but also model change (morphism).

- Sprinkle a few “advanced” keywords (anchors/links): math objects & morphism (only examples, no definitions).
... and day-dreaming ...

- “Vertical” specialization (in depth), went too far, leading to a general fragmentation of knowledge ... not seeing the forest anymore ...

- Alternative: slice mathematics (4 ex.) horizontally (breadth); this needs: better interfaces and more implementation layers, and social functions like math developers and project managers, open math programing and blogging as “DESC OK” etc.
What to Do?

Open our minds, **look around and NOW** (@21st century; & read more :) ), lead and mislead gently off the historically beaten path into the garden of wanders, down the rabbit hole of ...

Quantum (a.k.a Discrete) Mathematics ...

Less NUMBers ... MOoRE structures :) 

The End

... just another beginning :}


... more homework :)  

- Restructure curriculum:
  1) Regroup Math topics, like for e.g. Finite Mathematics (expand topics, including Finite Number Systems & Discrete Calculus)
  2) Import all “good” ideas into Discrete Math.

- Redesign the interface, bury the details and let the computers do the computations.
Appendix
Some historical “errors”

1) Assessment of breakthrough research: Galois, Armand (complex numbers) etc.
2) Naming theorems (Law of Eponymy)
3) Directions of development of math (e.g. Hilbert’s axiomatization problem etc.)
4) Ignoring Greek wisdom … (algebraic numbers)

etc.
$\mathbb{Z}_6$ as an arithmetic playground

- Show pictures (always): benzene (why not linking with sciences!?) … or their favorite hexagon (pizza box? etc)
- Label vertices: 0,1,2,3,4,5
- Play a game with dice: go around, +/-, x 2, x 3 … “vectors’=displacements, coordinates <-> modulo arithmetic (finite number systems) … WEB is the limit :)
Who’s LMI?

Mathematician, scientifically oriented, with computer science background and R&D experience; prefers music as an art …

etc.
A New Kind of Mathematics: A New Kind of Science (Stephen Wolfram) needs a New Kind of Mathematics:

1) **Simplify** by importing all “good” concepts in Discrete Mathematics (it’s happening :)

2) **Let computers compute** =)

3) **Restructure math details in layers**, with conceptual interfaces and implementations (hierarchy of details: from higher-2-lower level languages)